Queen Mary
University of London

Late-Summer Examination period 2017

MTH6127
Metric Spaces and Topology

Duration: 2 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks available are shown next to the questions.

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Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Examiners: M. Farber

In this examination the symbol $\mathbb{R}$ denotes the set of real numbers.

Question 1. [4 marks]

(a) Give the definition of *a metric space* (X,d). **[2]**

(b) Explain what it means for a subset $U \subseteq X$ in a metric space to be *open*. **[2]**

Question 2. [10 marks]

(a) When do we say that a sequence $\{x_n\}_{n\geqslant 1}$ of points in a metric space X *converges* to a point $x_0 \in X$? **[2]**

(b) When do we say that a sequence $\{x_n\}_{n\geqslant 1}$ of points in a topological space X *converges* to a point $x_0 \in X$? **[2]**

(c) Let X be a metric space. Is it possible that a sequence of points $\{x_n\}_{n\geqslant 1}$, $x_n \in X$ converges to two distinct points $x_0, x_0' \in X$, $x_0 \neq x_0'$? Justify your answer. **[2]**

(d) Consider $X = \mathbb{R}$ with the finite-complement topology (i.e. when open subsets are complements of the finite subsets). Consider the sequence $x_n = n \in X$ and find all points $x_0 \in X$ such that the sequence $\{x_n\}_{n\geqslant 1}$ converges to x_0. Justify your answer. **[4]**

Question 3. [10 marks]

(a) Explain what is meant for a metric space (X,d) to be *complete* and give an example of a metric space which is not complete. Justify your answer. **[2]**

(b) Let X be a metric space and let $F \subseteq X$ be a closed subset. Show that F is complete with respect to the induced metric. **[2]**

(c) Which of the following subsets of $\mathbb{R}$ are complete when considered as subspaces of $\mathbb{R}$ with the usual metric? Briefly justify your answer.

(i) $\{3^n; n = 1,2,\ldots\}$, **[2]**

(ii) $\{3^{-n}; n = 1,2,\ldots\}$, **[2]**

(iii) $\{3^{-n}; n = 1,2,\ldots\} \cup \{0\}$. **[2]**

Question 4. [10 marks]

(a) Define the sup metric on the set $C[0,\pi]$ of all real continuous functions on the closed interval $[0,\pi]$. **[3]**

(b) Is $C[0,\pi]$ complete? (No proof is required.) **[3]**

(c) Decide whether the sequence of functions

$$f_n(x) = \sin(nx), \quad x \in [0,\pi],$$

converges in $C[0,\pi]$ with respect to the sup metric. **[4]**

Question 5. [23 marks]

(a) Give the $\varepsilon - \delta$ definition of continuity of a map $f : X \to Y$ between metric spaces (X, d_X) and (Y, d_Y). **[3]**

(b) Show that if a map $f : X \to Y$ satisfies the $\varepsilon - \delta$ definition of continuity then for any open set $U \subseteq Y$ the preimage $f^{-1}(U) \subseteq X$ is open. **[4]**

(c) Give an example of a non-constant continuous map $f : \mathbb{R} \to \mathbb{R}$ and an open subset $U \subseteq \mathbb{R}$ such that the image $f(U) \subseteq \mathbb{R}$ is not open. **[5]**

(d) Show that if a map $f : X \to Y$ is continuous then for any closed set $F \subseteq Y$ the preimage $f^{-1}(F) \subseteq X$ is closed. **[4]**

(e) Is it true that the image of a closed set under a continuous map is closed? Explain your answer. **[7]**

Question 6. [17 marks]

(a) What is meant by *an open cover* of a topological space? **[2]**

(b) When do we say that a topological space is *compact*? **[3]**

(c) Which of the following subsets of the real line $\mathbb{R}$ are compact? Briefly justify your answer:

(i) $\mathbb{R}$; **[3]**

(ii) $[2,3]$; **[3]**

(iii) $(2,3)$; **[3]**

(iv) $[2,\infty)$. **[3]**

Question 7. [26 marks]

(a) State the contraction mapping theorem. [**5**]

(b) Consider $\mathbb{R}^2$ with the d_1-metric, i.e. $d_1(v,v') = |x-x'| + |y-y'|$ where $v = (x,y)$ and $v' = (x',y')$. Is this metric space complete? Justify your answer. [**5**]

(c) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f(v) = (\frac{1}{3}y, \frac{1}{2}(x+1)),$$

where $v = (x,y)$. Show that f is a contraction with respect to the d_1-metric. [**10**]

(d) Find the fixed point of f in part (c). [**6**]

End of Paper.