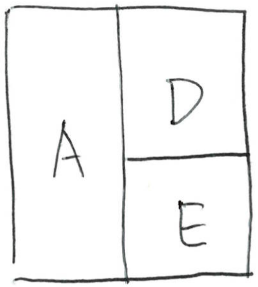
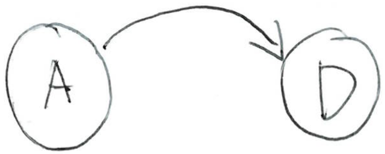
Week 10 Credit risk

structure



States



15.1 The Merton model

$F(t)$: value of a corporate entity at time t value = total capital $\simeq$ Asset

$$F(t) = E(t) + B(t)$$

$E(t)$: equity

$B(t)$: debt

Definition 15.5 (Merton Model) Slide 9+10

Assumptions:

1. At time $t=0$,

$$\underset{\text{value}}{F(0)} = \underset{\substack{\text{equity}\\ \text{shareholders}}}{E(0)} + \underset{\text{debt}}{B(0)}$$

$B(0)$: cost of the <u>zero coupon bonds</u> sold by the corporate entity

2. The corporate entity promises to pay to bondholders the amount L at future time T.

	0	T
Bondholder:	$-B(0)$ invest	L.

If $F(T) \geq L$: at time T the total value of the corporate entity is greater than or equal to ~~the~~ its debt L to bondholders

Bondholders: L.

Shareholders: $F(T) - L$

If $F(T) < L$: the corporate entity defaults

Bondholders: $F(T)$

Shareholders: 0

payoff: Shareholders: $R_{sh}(T) = \max\left[F(T) - L, 0\right] = \underline{(F(T) - L)^+}$

Bondholders: $R_{bh}(T) = \min\left[F(T), L\right] = F(T) - (F(T) - L)^+$

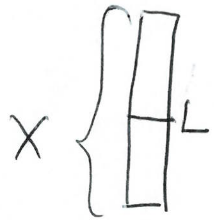
Remarks: $\min(x, y) = y - (y-x)^+ = x - (x-y)^+$

$L > B(0)$

Merton's observation

shareholders $R_{sh}(T) = \max[F(T) - L, 0] = (F(T) - L)^+$

European call option $Call(L, T)$



Lemma 15.1

Suppose interest rate: r

$$E(0) = e^{-rT} \tilde{E}(F(T) - L)^+$$

$\tilde{E}$: risk-neutral

Proof: Th 5.2 payoff function $R(T)$

price $C = e^{-rT}\tilde{E}(R(T))$

$R(T)$ for shareholders: $R_{sh}(T) = (F(T)-L)^+$

$C = e^{-rT}\tilde{E}(F(T)-L)^+$

Cost of the shares $= E(0) = C$

$E(0) = e^{-rT}\tilde{E}(F(T)-L)^+ \quad \blacksquare$

Remark: Model - independent $F(t)$

B-S formula $\rightarrow E(0)$

Th 15.1

$F(t) = F(0)e^{\mu t + \sigma W(t)} \sim GBM$

r

$E(0) = (E(0)+B(0))\Phi(w) - Le^{-rT}\Phi(w-\sigma\sqrt{T})$, $\quad w = \dfrac{\ln\frac{E(0)+B(0)}{L} + rT + \frac{1}{2}\sigma\sqrt{T}}{\sigma\sqrt{T}}$

Proof: Hint: $S = ?$ $K = ?$ $C = ?$

$$C = e^{-rT} \tilde{E}\left(F(T) - K\right)^{+} = S\,\Phi(w) - K e^{-rT}\,\Phi(w - \sigma\sqrt{T})$$

$$w = \frac{\ln\frac{S}{K} + rT}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}$$

$$\begin{cases} S = F(0) = E(0) + B(0) \\ K = L \\ C = E(0) \quad \text{cost of the shares} \end{cases}$$

$\boxed{}$

Corollary 15.1: Calculate $B(0)$ or $E(0)$ using Eq (*) Th 15.1

Corollary 15.2:

$$P(\text{default}) = P\left(F(T) < L\right)$$

CW W3-4

$\Phi(\quad)$

15.2 Two-state intensity-based model for credit ratings — reduced form model

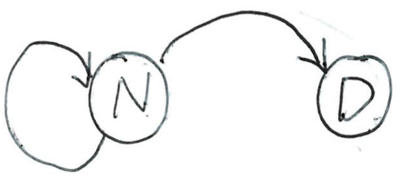
Definition 15.6

A two-state model assumes that:

- At each time t, a corporate entity can be in one of two states.
  - N = not previously defaulted
  - D = defaulted

$X(t)$: the state of the corporate entity at time t.

$X(t) = N$ or $X(t) = D$



Definition 15.7

- There is a function $\lambda(t) \geq 0$ s.t. for $\Delta t > 0$

$$P\left(X(t+\Delta t) = N \mid X(t) = N\right) = 1 - \lambda(t)\Delta t + o(\Delta t) \qquad (6)$$

$$P\left(X(t+\Delta t) = D \mid X(t) = N\right) = \lambda(t)\Delta t + o(\Delta t)$$

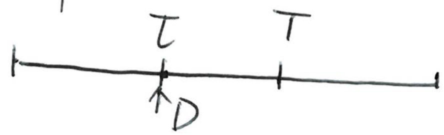
$$\lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0$$

$\lambda(t)$: <u>transition intensity</u> from N to D.

- τ : the time of default

$$\text{payment} = \begin{cases} 1 & \text{if } \tau > T \quad (\text{no default by time } T) \\ \delta & \text{if } \tau \leq T \quad (\text{default takes place by time } T) \end{cases}$$

$0 \leq \delta < 1$



Prob of default　　distribution of the time of default

Default takes place if $\tau \leq T$

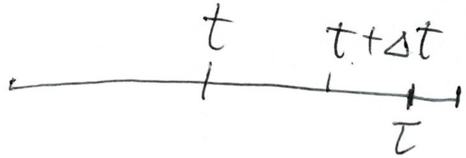
$$P(\text{default}) = \mathbb{P}(\tau \leq T)$$

$$F_\tau(t) = P(\tau \leq t) \leftarrow \text{CDF of the time of default } \tau$$

Theorem 15.2

For $t \geq 0$, $\quad F_\tau(t) = 1 - e^{-\int_0^t \lambda(s)\,ds}$

Lemma 15.2



$$p'(t) = -\lambda(t)\,p(t)$$

Proof 15.2

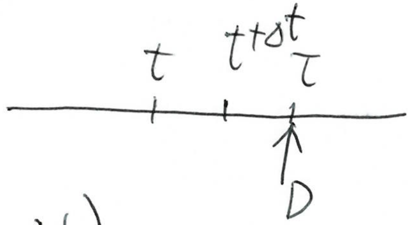
$$\Delta t > 0, \quad p(t+\Delta t) = \mathbb{P}(\underbrace{\tau > t+\Delta t}_{A}) \overset{(*)}{=} \mathbb{P}(\underbrace{\tau > t+\Delta t}_{A} \text{ and } \underbrace{\tau > t}_{B})$$

$(*)$ $\quad \mathbb{P}(A) = \mathbb{P}(A \cap B)$ if $A \subset B$

$$\overset{(**)}{=} \mathbb{P}(\underbrace{\tau > t}_{B})\,\mathbb{P}(\underbrace{\tau > t+\Delta t}_{A} \mid \underbrace{\tau > t}_{B})$$

$$(**):\quad \mathbb{P}(A \cap B) = \mathbb{P}(B)\,\mathbb{P}(A \mid B)$$

$$A = \{\tau > t+\Delta t\},\quad B = \{\tau > t\}$$



$$p(t+\Delta t) = p(t)\,\mathbb{P}(\tau > t+\Delta t \mid \tau > t)$$

$$= p(t)\,\mathbb{P}\big(X(t+\Delta t) = N \mid X(t) = N\big)$$

$$= p(t)\big(1 - \lambda(t)\Delta t + o(\Delta t)\big) \quad \text{using Eq (6)}$$

$$p(t+\Delta t) = p(t)\big(1 - \lambda(t)\Delta t + o(\Delta t)\big) = p(t) - \lambda(t)p(t)\Delta t + o(\Delta t$$

$$\frac{p(t+\Delta t) - p(t)}{\Delta t} = -\lambda(t)p(t) + \frac{o(\Delta t)}{\Delta t}$$

$$\Delta t \to 0 \qquad p'(t) = \lim_{\Delta t \to 0} \frac{p(t+\Delta t) - p(t)}{\Delta t} = -\lambda(t)p(t)$$

$$p'(t) = -\lambda(t)p(t) \qquad \blacksquare$$

Proof of Th 15.2:

$p(t) = \mathbb{P}(\tau > t)$, $F_\tau(t) = 1 - p(t)$

survival prob

$F_\tau(t) = \mathbb{P}(\tau \le t)$

CDF

target: $P(\tau > t) = \boxed{p(t) = e^{-\int_0^t \lambda(s)\,ds}}$

Use Lemma 15.2 $\quad p'(t) = -\lambda(t)\,p(t)$

$$\frac{p'(t)}{p(t)} = -\lambda(t), \qquad \left[\ln(p(t))\right]' = -\lambda(t)$$

$$\int_0^t \ln(p(s))'\,ds = -\int_0^t \lambda(s)\,ds$$

$$\ln(p(t)) - \ln(p(0)) = -\int_0^t \lambda(s)\,ds$$

$$\frac{p(t)}{p(0)} = \exp\left(-\int_0^t \lambda(s)\,ds\right) \quad \text{so} \quad \boxed{p(t) = p(0)\,e^{-\int_0^t \lambda(s)ds}}$$

$p(0) = 1 \qquad p(0) = \mathbb{P}(\tau > 0)$ → time of defult

∎

Collollary 15.3

$$\mathbb{P}(\text{default}) = \mathbb{P}(\tau \le T) = 1 - e^{-\int_0^T \lambda(s)\,ds}$$

CDF $\quad F_\tau(T)$

Remarks: 1. $F_\tau(t) = 0$ if $t < 0$

$F_\tau(t)$ if $t \geq 0$ $\qquad p(0) = 1$

At the beginning $t=0$, not defaulted yet

2. λ is a constant $\quad \lambda \perp t$

$F_\tau(t) = 1 - e^{-\lambda t}$ CDF

$f_\tau(t) = F'(t) = \boxed{\lambda e^{-\lambda t}}$ PDF $\qquad$ if $t \geq 0$

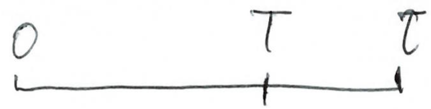
$\tau \sim \text{Exp}(\lambda)$

3. $\mathbb{P}(\text{default}) \to 1$ iff $\int_0^\infty \lambda(s)\,ds = \infty$

Apply to bonds:

$B(t,T)$: RN price at time t, $0 \le t \le T$, T term of the bond

$R(T)$: payoff function



$$R(T) = \begin{cases} 1 & \text{if } \tau > T \text{ no default} \\ \delta & \text{if } \tau \le T \text{ default} \end{cases}$$

Q: $B(t,T)$?

A: Th5.2 $B(t,T) = e^{-rT}\tilde{E}(R(T))$

$\tilde{\lambda}(t)$ risk-neutral intensity exits

$\mathbb{P}(\text{default}) = \tilde{P}(\tau \le T) = 1 - e^{-\int_0^T \tilde{\lambda}(s)\,ds}$ ~~Cott~~ Corollary 15.3

$\tilde{P}(\tau > T) = e^{-\int_0^T \tilde{\lambda}(s)\,ds}$

$$\tilde{E}(R(T)) = 1 \times \tilde{\mathbb{P}}(\underbrace{R(T)=1}_{\text{no default}}) + \delta \times \tilde{\mathbb{P}}(R(T)=\delta)$$

$$= 1 \times \tilde{\mathbb{P}}(\tau > T) + \delta \times \tilde{\mathbb{P}}(\tau \leq T)$$

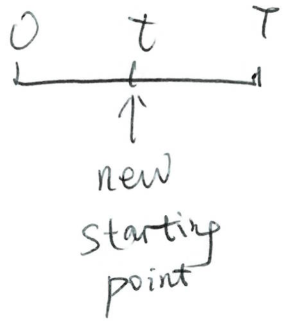
$$= 1 \times e^{-\int_0^T \tilde{\lambda}(s)\,ds} + \delta \times \left(1 - e^{-\int_0^T \tilde{\lambda}(s)\,ds}\right)$$

$$= (1-\delta)\, e^{-\int_0^T \tilde{\lambda}(s)\,ds} + \delta$$

$$B(0,T) = e^{-rT}\,\tilde{E}(R(T)) = e^{-rT}\left((1-\delta)\, e^{-\int_0^T \tilde{\lambda}(s)\,ds} + \delta\right)$$

$t=0$

$$B(t,T) = e^{-r(T-t)}\left((1-\delta)\, e^{-\int_t^T \tilde{\lambda}(s)\,ds} + \delta\right)$$



15.3 The Jarrow-Lando-Turnbull (JLT) model

$X(t)$: rating of the bond at time t

$$p_{ij}(s,t) = \mathbb{P}\left(X(t)=j \mid X(s)=i\right), \quad \text{where } 1 \leq i,j \leq n;\ s \leq t$$

i, j: states / ratings

s: starting time

t: ending time

Definition 15.9 JLT

$$P_{ij}(t, t+\Delta t) = \lambda_{ij}(t)\Delta t + o(\Delta t) \text{ if } i \neq j$$

$\cancel{P_{ij}}$

$$P_{ii}(t, t+\Delta t) = 1 - \lambda_{ii}(t)\Delta t + o(\Delta t)$$

$\lambda_{ij}(t) \geq 0$ are the transition intensities

$$\lambda_{ii}(t) = \sum_{\substack{1 \leq j \leq n, \\ j \neq i}} \lambda_{ij}(t)$$

Markov chains

1 2 . . . . $\cancel{i}$ . . . $\cancel{j}$ . . . n

i j

$$\sum_{j=1}^{n} P_{ij}(s,t) = 1$$