Queen Mary
University of London

Main Examination period 2021 – May/June – Semester B
Online Alternative Assessments

MTH5131: Actuarial Statistics – SOLUTIONS

All questions are unseen, but similar to lectures or coursework, except 5 which is just unseen.

Question 1 [12 marks]. [U]

(a) Previous national household longitudinal studies could be used for pre-lockdown data. They could also be used for post-lockdown data, if they exist, or a new study could be carried out. **[3]**

(b) This is an inferential study. **[3]**

(c) This study is longitudinal in nature. You would want to know how the alcohol, e-cigarette and cigarette consumption of respondents to the study over a timespan that includes the lockdown. **[3]**

(d) You might plot alcohol, e-cigarette and cigarette consumption over a timespan that includes the lockdown among different categories of respondents. **[3]**

Total marks so far: 12
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Question 2 [9 marks].

(a) We solve the equation

$$S = \begin{pmatrix} 5 & 2 & 0 \\ 2 & 6 & 2 \\ 0 & 2 & 7 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 9 \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow \begin{cases} -4a + 2b = 0 \\ 2a - 3b + 2c = 0 \\ 2b - 2c = 0 \end{cases}$$

One solution is

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

We normalise to give

$$\begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}$$

The other possible answer is

$$\begin{pmatrix} -1/3 \\ -2/3 \\ -2/3 \end{pmatrix}$$

[6]

(b) The eigenvalues all lie on a line in the Skree diagram ansd so they are all on the mountain.

The two largest eigenvalues only take $15/18 = 0.833 < 0.90$ of the total variance. Either way, all components are principal. **[3]**

Total marks so far: 21

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Question 3 [16 marks].

(a) (i) The data is already ranked (although not in order from smallest to largest). The differences are We find that $\sum_i d_i^2 = 22$ and so

Candidate	Interviewer 1	Interviewer 2	Difference
A	7	8	-1
B	2	3	-1
C	10	9	1
D	14	14	0
E	15	13	2
F	6	5	1
G	1	2	-1
H	8	7	1
I	9	10	-1
J	4	4	0
K	5	6	-1
L	3	1	2
M	13	15	-2
N	11	12	-1
O	12	11	1

$$r_s = 1 - \frac{6 \times 22}{15 \times (15^2 - 1)} = 0.9607$$

[5]

(ii) The t-statistic with $15 - 2 = 13$ degrees of freedom is

$$\frac{(0.9607)\sqrt{13}}{\sqrt{1 - 0.9607^2}} = 12.48077$$

The p-value $2P(Y > 12.48077)$ is much less than 1% and is strong evidence to conclude that the rankings are correlated.

[4]

(b) We make a table of concordant and discordant pairs:

Rank1	Rank2	C	D
1	2	13	1
2	3	12	1
3	1	12	0
4	4	11	0
5	6	9	1
6	5	9	0
7	8	7	1
8	7	7	0
9	10	5	1
10	9	5	0
11	12	3	1
12	11	3	0
13	15	0	2
14	14	0	1
15	13		

Totalling the columns gives $n_c = 96$ and $n_d = 9$ Thus

$$\tau = \frac{96 - 9}{(15)(14)/2} = 0.82857$$

[7]

Total marks so far: 37

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Question 4 [18 marks].

(a) The density of Y is

$$f_Y(y) = \theta y^{\theta-1}, \quad 0 < y < 1.$$

The first moment is

$$E(Y) = \int_0^1 y f_Y(y)\,dy = \int_0^1 \theta y^\theta\,dy = \frac{\theta}{\theta+1}$$

We set

$$\overline{y} = E(Y) = \frac{\theta}{1+\theta} \Rightarrow \tilde{\theta} = \frac{\overline{y}}{1-\overline{y}}$$

We find that

$$\overline{y} = \frac{0.92 + 0.79 + 0.90 + 0.65 + 0.86 + 0.47 + 0.73 + 0.97 + 0.94 + 0.77}{10} = 0.8$$

Therefore

$$\tilde{\theta} = \frac{0.8}{1-0.8} - 1 = 4.$$

[6]

(b) The likelihood is

$$\theta^{10}\left(\prod_{i=1}^{10} y_i\right)^{\theta-1}.$$

The log-likelihood is

$$\ell = 10\ln(\theta) + (\theta-1)\sum_{i=1}^{10}\ln(y_i).$$

Taking derivatives:

$$\frac{10}{\theta} + \sum_{i=1}^{10}\ln(y_i) = 0 \Rightarrow \hat{\theta} = -\frac{10}{\sum_{i=1}^{10}\ln(y_i)}$$

(Clearly, the second derivative is negative.) We get

$$\hat{\theta} = 4.116$$

[5]

(c) The likelihood is

$$\theta^{10}\left(\prod_{i=1}^{10} y_i\right)^{\theta-1} (P(Y < 0.60))^2$$

We calculate that

$$P(Y < 0.60) = \int_0^{0.60} \theta y^{\theta-1} = (0.60)^\theta$$

The log-likelihood is

$$\ell = 10\ln(\theta) + (\theta - 1)\sum_{i=1}^{10}\ln(y_i) + 2\theta\ln(0.60)$$

Taking derivatives:

$$\frac{10}{\theta} + \sum_{i=1}^{10}\ln(y_i) + 2\ln(0.60) = 0 \Rightarrow \hat{\theta} = -\frac{10}{\sum_{i=1}^{10}\ln(y_i) + 2\ln(0.60)}$$

(Clearly, the second derivative is negative.) We get

$$\hat{\theta} = 2.8976$$

The decrease is the in the maximum likelihood estimator estimator is due to the extra information that some data were small relative to the other data. **[7]**

Total marks so far: 55

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Question 5 [12 marks].

(a) The expectation of th estimator is

$$\begin{aligned} E[C(Y_1 + Y_2 + Y_3)] &= C\left(E[Y_1] + E[Y_2] + E[Y_3]\right) \\ &= C\left(\frac{1}{\alpha} + \frac{1}{2\alpha} + \frac{1}{4\alpha}\right) \\ &= \frac{7C}{4\alpha}. \end{aligned}$$

In order for $C(Y_1 + Y_2 + Y_3)$ to be unbiased for $1/\alpha$ we must have

$$C = \frac{4}{7}$$

[7]

(b) The MSE of the the unbiased estimator we found in (a) is its variance:

$$C^2\left(\frac{1}{\alpha^2} + \frac{1}{4\alpha^2} + \frac{1}{16\alpha}\right)$$

$$= C^2\frac{21}{16\alpha^2} = \frac{16}{49} \times \frac{21}{16\alpha^2} = \frac{3}{7\alpha^2}$$

[5]

Total marks so far: 67

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Question 6 [9 marks].

(a) The prior is

$$f(p) \propto p^{4-1}(1-p)^{3-1} = p^3(1-p)^2$$

The likelihood is

$$f(y|p) \propto \prod_{i=1}^{10}(p(1-p)^{y_i}) = p^{10}(1-p)^{\sum_{i=1}^{n} y_i} = p^{10}(1-p)^{42}$$

The posterior is proportional to

$$p^3(1-p)^2 \times p^{10}(1-p)^{42} = p^{13}(1-p)^{44}$$

Therefore, the posterior is $\text{Beta}(14, 45)$ distributed. **[6]**

(b) The Bayesian estimate of p under square error loss is the expectation of the posterior. The posterior is $\text{Beta}(14, 45)$. Therefore, the Bayesian estimate of p under squared error loss is

$$\frac{14}{14+45} = \frac{14}{59}$$

[3]

Total marks so far: 76

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Question 7 [9 marks].

(a) $E[s^2(\theta)]$ is estimated by the average of the sample variances:

$$E[s^2(\theta)] = \frac{130 + 60 + 35 + 100}{4} = 81.25$$

The sample mean of the $\overline{X_i}$'s is:

$$\overline{X} = \frac{125 + 85 + 140 + 175}{4} = 131.25$$

and the sample variance of the X_i's is:

$$\begin{aligned} & \frac{1}{4-1}\sum_{i=1}^{3}(\overline{X_i} - \overline{X})^2 \\ = & \frac{(125-131.25)^2 + (85-131.25)^2 + (140-131.25)^2 + (175-131.25)^2}{3} \\ = & 1389.583 \end{aligned}$$

Moreover,

$$\text{var}[m(\theta)] = \frac{1}{3}\sum_{i=1}^{3}(\overline{X_i} - \overline{X})^2 - \frac{1}{5}E[s^2(\theta)] = 1389.583 - \frac{1}{5} \times 81.25 = 1373.333$$

The credibility factor is

$$Z = \frac{5}{5 + \frac{81.25}{1373.333}} = 0.9883$$

[6]

(b) The credibility estimate of the amount per claim for the coming year for product 3 is

$$(1 - 0.9883) \times 131.25 + 0.9883 \times 140 = 139.8977$$

[3]

Total marks so far: 85

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Question 8 [15 marks].

(a) In parameterised form, the linear predictors are (with i corresponding to the levels of YO):

$$\text{Model 1}: \quad \alpha_i \quad (2 \text{ parameters})$$

There are two parameters for the 2 combinations of YO.

$$\text{Model 2}: \quad \alpha_i + \beta_j \quad (4 \text{ parameters})$$

There is one parameter to set the base level for the combination YO_0, FMS_0 and three additional parameters for the combinations of the higher levels of the two factors.

[6]

(b) The completed table, together with the differences in the scaled deviance and degrees of freedom, is shown below.

Model	Scaled Deviance	Degrees of Freedom	Δ Scaled Deviance	Δ Degrees of Freedom
1	40	5		
YO	30	4	10	1
$YO + FMS$	23	2	7	2

(It is not necessary for the students to add the additional columns to the table.)

Comparing the constant model and Model 1

The difference in the scaled deviances is 10.

This is greater than 3.841, the upper 5% point of the χ^2_1 distribution.

So Model 1 **is** a significant improvement over the constant model.

Alternatively, if we use the AIC to compare models, we find that since Δ(deviance)$> 2 \times \Delta$ degrees of freedom, because $10 > 2 \times 1$, Model 1 **is** a significant improvement over the constant model.

Comparing Model 1 and Model 2

The difference in the scaled deviances is 7.

This is greater than 5.991, the upper 5% point of the χ^2_2 distribution.

So Model 2 **is** a significant improvement over Model 1.

Alternatively, if we use the AIC to compare models, we find that since Δ(deviance)$> 2 \times \Delta$ degrees of freedom, because $7 > 2 \times 2$, Model 2 **is** a significant improvement over model 1.

[9]

Total marks so far: 100

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End of Paper.