MTH 4104 Example Sheet V Solutions

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V-1. (a)

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 5 & 8 & 2 & 9 & 6 & 10 & 7 & 11 & 1 & 4 & 3 \end{pmatrix}$$

(b)

$$h^{-1} = (11\,6\,7\,9\,3\,10\,1)(8\,2)(5\,4),$$

$$g \circ h = (1\,4\,6\,3)(2\,11\,5\,9\,7\,10),$$

$$h^{-1} \circ g \circ h = (1\,5\,3\,11\,4\,7)(2\,6\,10\,8).$$

(c) The order of a cycle is the length of the cycle. The order of an element g in S_n is the LCM of the orders of the cycles in g. With these in mind, the order of g is $\mathrm{lcm}(6,4) = 12$ and the order of $h^{-1} \circ g \circ h$ is $\mathrm{lcm}(6,4) = 12$.

In fact, $(h^{-1} \circ g \circ h)^r = (h^{-1} \circ g \circ h) \circ \cdots \circ (h^{-1} \circ g \circ h) = h^{-1} \circ g^r \circ h$. It therefore follows that $g^r = 1$ if and only if $(h^{-1} \circ g \circ h)^r = h^{-1} \circ g \circ h = 1$, i.e. the order of g equals the order of $h^{-1} \circ g \circ h$.

V-2. If g is a permutation in S_n of order r, then it is a product of cycles of length $\ell_1, \ldots, \ell_s$ (all $\geqslant 1$), such that $\ell_1 + \cdots + \ell_s = n$ and $\mathrm{lcm}(\ell_1, \ldots, \ell_s) = r$. If no such integers exist, there is no permutation of order s. If, on the other hand, such $\ell_1, \ldots, \ell_r$ exist, then it might be possible (not guaranteed!) that a permutation of order s exists. (a) For LCM to be 14, there has to be a cycle divisible by 7. If the order is 7, then there can be one more cycle of order 1 only (since $n = 8$) and $\mathrm{lcm}(7, 1) = 7$, not 14. Hence there is no permutation of order 14. (b) It seems $\ell_1 = 5, \ell_2 = 3$ define a possible cycle type. Indeed $(1\,2\,3\,4\,5)(6\,7\,8)$ is an example. (c) Since $\mathrm{lcm}(\ell_1, \ldots, \ell_s) = 2^4$, any one of $\ell_1, \ldots, \ell_s$ would be a power of 2. In fact, there has to be a cycle of order 2^4. However, $2^4 > 8$, hence there cannot be a permutation of order 16.

V-3. No. $(a * b) * c = d * c = a$ but $a * (b * c) = a * d = c$.

V-4. We check the group axioms. (G0) If x and y are integers, then so is $x * y = x + y + 1$. (G1) $(x * y) * z = (x + y + 1) * z = x + y + 1 + z + 1$. On the other hand, $x * (y * z) = x * (y + z + 1) = x + y + z + 1 + 1$. They are equal. (G2) We must find an integer e which satisfies $x * e = e * x = x$, i.e. $x + e + 1 = x$. So e is inevitably -1. Indeed, $x * (-1) = x + (-1) + 1 = x$ and $(-1) * x = (-1) + x + 1 = x$. (G3) Given x, we must find y such that $x * y = -1$, i.e. $x + y + 1 = -1$, i.e. $y = -x - 2$. Indeed, $x * (-x - 2) = x - x - 2 + 1 = -1 (= e)$ and $(-x - 2) * x = -x - 2 + x + 1 = -1 (= e)$.