Week 11

Plan: (1) GW's from slow, compact sources

(2) Frequency domain and multipole expansion

(3) Gravitational binaries (as GW sources)

(1) Working in the linearised theory we can identify T_{ab}, the r.h.s of the last Eq. on week 10 notes, with the stress tensor of the matter. Assuming that we deal with a compact far-away source, $T_{ab} = 0$ at the location of the observer (far region). At the location of the source, $T_{ab} \neq 0$ and we can solve our equation by using the Green function of $\partial_a \partial_b \eta^{ab}$

$$\bar{h}_{ab}(t, \underline{x}) = 16\pi G \int d^3x' \frac{1}{4\pi} \frac{T_{ab}(ct - |\underline{x} - \underline{x}'|, \underline{x}')}{|\underline{x} - \underline{x}'|}$$

We then would like to connect the near-region solution to the far-away $\bar{h}_{ab}$ which is detected. At the linear order it is sufficient to take the limit of $\bar{h}_{ab}(t, \underline{x})$ for t, $r = |\underline{x}| \to \infty$ with $u = t - r$ kept fixed (u is called retarded time).

We will make another approximation (beside focusing on linear effects): we consider slow sources in comparison with the speed of light. Thus we have

$$T_{ab}(ct-|\underline{x}-\underline{x}'|,\underline{x}') \simeq T_{ab}(u,\underline{x}') + (|\underline{x}|-|\underline{x}-\underline{x}'|)\,\partial_0 T_{ab}(u,\underline{x}')$$

More explicitly $|\underline{x}-\underline{x}'| = (\underline{x}^2 - 2\underline{x}\,\underline{x}' + \underline{x}'^2)^{1/2} \xrightarrow[\text{region}]{\text{far}}$

$$|\underline{x}|\left(1 - \frac{2\underline{x}\cdot\underline{x}'}{|\underline{x}|^2} + \dots\right)^{1/2} \simeq |\underline{x}| - \frac{\underline{x}\cdot\underline{x}'}{|\underline{x}|} + \dots \Rightarrow$$

$$T_{ab}(ct-|\underline{x}-\underline{x}'|,\underline{x}') \simeq T_{ab}(u,\underline{x}') + \frac{\underline{x}\cdot\underline{x}'}{|\underline{x}|}\,\partial_0 T_{ab}(u,\underline{x}') + \dots$$

In the slow case the ratio between the second and the first term is small

$$\underbrace{\left(\frac{\underline{x}}{|\underline{x}|}\right)\cdot\underline{x}'}_{\substack{\text{size of}\\ \text{the source}\\ |\underline{x}'|<d}} \;\; \underbrace{\frac{\partial_0 T_{ab}}{T_{ab}}}_{\sim \frac{1}{cT}} \Rightarrow \underbrace{\left(\frac{d}{c}\right)}_{\text{"crossing time"; i.e. how long light}} \cdot \frac{1}{T} \ll 1$$

where T is the timescale of the variation of T_{ab} → related to the GW frequency

Then for the spatial component of h we have

$$\bar{h}_{ij}(u,r) \approx \frac{4G}{r}\int d^3x'\, T_{ij}(u,x') \qquad u = t-r \; [= t']$$

$$= \frac{2G}{r}\int d^3x' \Big[\partial_k (T_{ik}\, x'_j) - (\partial_k T_{ik})\, x'_j +$$
$$\partial_k (T_{jk}\, x'_i) - (\partial_k T_{jk})\, x'_i \Big]$$

By evaluating the derivatives and using $\partial_i x_j = \delta_{ij}$, one can check that each line is equal to T_{ij}. We follow another way: the first term in each line is a total derivative and can be neglected and we can use the conservation law

$\partial_b T_a{}^b = 0 \overset{a=i}{\Rightarrow} \partial_0 T_i{}^0 + \partial_k T_i{}^k = 0$. Then

$$\bar{h}_{ij} = \frac{2G}{r}\int d^3x' \Big[(\partial_0 T_i{}^0)\, x'_j + (\partial_0 T_j{}^0)\, x'_i \Big] =$$

$$\frac{2G}{r}\partial_0 \Big\{ \int d^3x'\, \partial_k (T_k{}^0\, x'_i x'_j) - (\partial_k T_k{}^0)\, x'_i x'_j \Big\}$$

Again neglecting total derivatives and using the conservation law above for $a=0$, we have

$$\bar{h}_{ij} = \frac{2G}{r}\partial_0^2 \Big\{ \int d^3x'\, T^{00} x'_i x'_j \Big\}$$

The curly parenthesis is often called I_{ij}

$$I_{ij}(v) = \int d^3x' \, T^{00} x'_i x'_j$$

What about the other components of $\bar{h}$?

Recall the condition we imposed on h while deriving the linearised wave equation: $\partial^a \left(h_{ab} - \frac{1}{2}\eta_{ab} h\right) = 0$. It does not fix h_{ab} completly as we can still perform change of coordinates parametrised by ξ with

$$\partial^a \left(\delta \bar{h}_{ab}\right) = \partial^a \left(\partial_a \xi_b + \partial_b \xi_a - \frac{1}{2}\eta_{ab} 2 \partial_c \xi^c\right) = 0$$

This is satisfied provided $\partial_a \partial^a \xi_b = 0$ since the first and the last term of the parenthesis cancel.

For instance we can use $\xi_a = X_a \, e^{i k_c x^c}$ with $k^2 = 0$

Let's see how it works: since $\bar{h}$ in the far zone is a solution of the free wave equation, it is a linear combination of plane waves

$$\bar{h}_{ab} \sim \sum_k \mathrm{Re}\left(H_{ab} \, e^{i k_c x^c}\right)$$

By focusing on one term in the sum, we

can choose a frame where $\kappa = \omega(1, 0, 0, 1)$. Then

$$\delta H_{ab} = i\left(\kappa_a X_b + \kappa_b X_a - \eta_{ab} \kappa_c X^c \right) \Rightarrow$$

$$\delta H_{0j} \sim i X_j \text{ with } j = 1, 2; \quad \delta H_{00} = \delta H_{03} \sim i\left(X_0 + X_3 \right)$$

We can choose X so that δH above cancels $\bar{h}_{01}$, $\bar{h}_{02}$, $\bar{h}_{00} + \bar{h}_{03}$. The combination $\bar{h}_{00} - \bar{h}_{03}$ vanishes because $\bar{h}_{ab}$ is harmonic

$$\partial^a \bar{h}_{ab} = 0 \overset{b=0}{\Rightarrow} \kappa^0 H_{00} + \kappa^3 H_{30} \sim \left(-H_{00} + H_{30} \right) = 0$$

Similarly by using $b = 1, 2, 3$ in the gauge condition after killing $\bar{h}_{0a}$ (recall that the gauge condition is preserved) we see that $\bar{h}_{13} = \bar{h}_{23} = \bar{h}_{33} = 0$.

So the transformation above can be used to kill all components $\bar{h}_{0a}$ and $\bar{h}_{3a}$ leaving only

$$H_{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & H_{11} & H_{12} & 0 \\ 0 & H_{12} & H_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow H_{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & H_+ & H_X & 0 \\ 0 & H_X & -H_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

since $\partial^a \bar{h}_{ab} = 0$ implies for the H_{ab} on the left

$\bar{h} = 0$ so H_{ab} is traceless.

[gauge or coordinates independent]

Punchline: there are two independent, physical quantities in the gravitational wave: H_+, $H_\times$. The symbol refers to the type of spacetime deformation they induce, see Fig. 7.1 of the LaTeX notes.

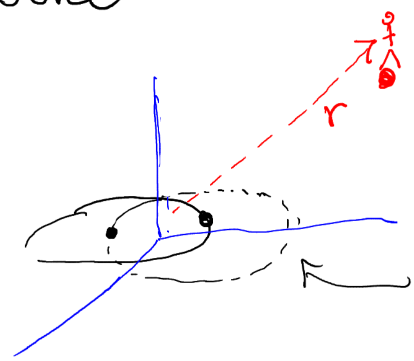
(2) The wave $\bar{h}_{ab}$ has been introduced as a function of the retarded time u, but at the end of pag. 4 we wrote it in terms of a vector k_c. The precise relation between these two ways of writing things is via a Fourier transform

$$\bar{h}_{ab}(\underline{x}) = \int \frac{d\omega}{2\pi} e^{-i\omega u} W_{ab}(\omega, \hat{n}) \quad \text{and}$$

$$W_{ab}(\omega, \hat{n}) = \int du \, e^{i\omega u} \bar{h}_{ab}(\underline{x})$$

where $\bar{h}_{ab}$ is the result at "null infinity" (i.e. $t, r \to \infty$ with u fixed and $\hat{n} = \frac{x}{r}$). W_{ab} is the waveform in the frequency domain. Current detectors are sensitive to GW's in the band $\sim 10\,Hz - 10^4\,Hz$), so it is important to understand the frequency of the GW's emitted by different sources.

The wave depends also on the relative orientation $\hat{x}$ between the source and the observer – Keep this picture in mind



$$\hat{n} = (\sin\vartheta \cos\varphi,\ \sin\vartheta \sin\varphi,\ \cos\vartheta)$$

Binary of schwarzschild BHs

The angular dependence can be expanded in term of multipoles, i.e. polynomials in $\hat{n}$. This is particularly useful for slow sources since higher multipoles (with k factors of $\hat{n}$) are small with respect to the leading one (by a factor of $\left(\frac{v}{c}\right)^k$). Focusing on the leading term

$$\bar{h}_{ij} = \frac{4G}{r} P_{ijkl} \left[\frac{1}{2} U_{\langle kl \rangle} + \dots \right]$$

terms with n^k, $k \geq 1$

where $U_{\langle kl \rangle}(u)$ is symmetric-traceless in k, l and depends only on u. The last ingredient needed is

$$P_{ijkl} = P_{ik} P_{jl} - \frac{1}{2} P_{ij} P_{kl}, \qquad P_{ij} = \delta_{ij} - \hat{n}_i \hat{n}_j$$

which is a projector from $\mathbb{R}^3$ to the space of physical polarisations (since $\hat{n}_i \hat{n}^i = 1$ as you can check).

(3) As already mentioned, we'll focus on binary systems as source of GWs. Given our assumption of small velocities ($v/c \ll 1$) it makes sense to study these systems in the Newtonian approximation. In the tutorial 10, we saw how to separate the relative motion from that of the centre of mass. So let us work in a frame where $P_{C.M.} = 0$ and focus on the Lagrangian of the relative motion

$$L = \frac{1}{2}\mu \dot{\underline{x}}^2 + \frac{GM\mu}{|\underline{x}|} \quad \text{with} \quad \begin{array}{l} \underline{X} = \underline{x}_1 - \underline{x}_2 \\ M = m_1 + m_2 \ \& \ \mu = \dfrac{m_1 m_2}{M} \end{array}$$

It is easy to show that the angular momentum

$$L_i = (\underline{x} \times \underline{p})_i = \epsilon_{ijk} x^j p^k \quad \text{with} \quad p^k = \frac{\partial L}{\partial \dot{x}^k} \quad \text{is conserved}$$

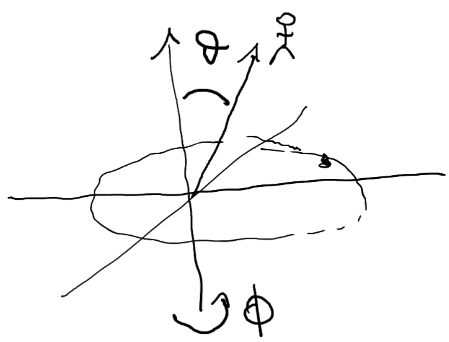
$$\frac{d}{dt}(\underline{x} \times \underline{p})_i = \epsilon_{ijk}\Big(\underbrace{\dot{x}^j p^k}_{(a)} + \underbrace{x^j \frac{dp^k}{dt}}_{(b)} \Big)$$

The first term (a) vanishes since $p^k = \mu \dot{x}^k$ and the antisymmetric Levi-Civita symbol is contracted with $\dot{x}^j \dot{x}^k$. For the second term (b) we use the Euler-Lagrange equation $\dfrac{dp^k}{dt} = \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{x}^k} = \dfrac{\partial L}{\partial x^k} = -\dfrac{G\mu M x^k}{|\underline{x}|^3}$

and again it vanishes for the same reason.

Comment: this is a consequence of the potential being just a function of $|\underline{x}|$ and the same property holds for any $V(|\underline{x}|)$.

Since L_i is constant the motion takes place in the plane orthogonal to $\underline{L}$ and it is convenient to introduce spherical coordinates (r, θ, φ) so that this motion takes place in the equatorial plane



$\Rightarrow$ The trajectory has $\vartheta = \frac{\pi}{2}$ and the Lagrangian can be simplified as follows

$$L = \frac{1}{2}\mu\left[\dot{\rho}^2 + \rho^2\left(\dot{\vartheta}^2 + \sin^2\vartheta\,\dot{\varphi}^2\right)\right] + \frac{GM\mu}{\rho}$$

$$\overset{\vartheta = \pi/2}{=} \frac{1}{2}\mu\left[\dot{\rho}^2 + \rho^2\dot{\varphi}^2\right] + \frac{GM\mu}{\rho}$$

Then $\frac{\partial L}{\partial \dot{\varphi}} = \mu\rho^2\dot{\varphi} = L \leftarrow$ constant. The associated Hamiltonian (the energy) is $H = T + V$

$$T + V = \frac{1}{2}\mu\left[\dot{\rho}^2 + \rho^2\dot{\phi}^2\right] - \frac{GM\mu}{\rho}$$

which in terms of conserved quantities reads

$$H = \frac{p_\varrho^2}{2\mu} + \underbrace{\frac{L^2}{2\mu\varrho^2} - \frac{GM\mu}{\varrho}} \quad \text{with } p_\varrho = \mu\dot\varrho$$

Thus the effective potential is $V_{eff}(\varrho)$

Notice that H does not depend explicitly on t which implies that H is constant. Thus if $T+V=E$ we get

$$E = \frac{1}{2}\mu\dot\varrho^2 + \frac{L^2}{2\mu\varrho^2} - \frac{GM\mu}{\varrho} \Rightarrow \frac{d\varrho}{dt} = \pm\left(\frac{2}{\mu}\left(E + \frac{GM\mu}{\varrho}\right) - \frac{L^2}{\mu^2\varrho^2}\right)^{1/2}$$

$$L = \mu r^2\dot\varphi \Rightarrow \frac{d\varphi}{d\varrho} = \frac{d\varphi}{dt}\frac{1}{\frac{d\varrho}{dt}} = \pm\left(\frac{2}{\mu}\left(E + \frac{GM\mu}{\varrho}\right) - \frac{L^2}{\mu^2\varrho^2}\right)^{-1/2}\frac{L}{\mu\varrho^2}$$

These equations can be integrated in terms of trigonometric function (the smoking gun is that the polynomial under the square root is quadratic).

Let us check that the following trajectory solves our problem in the case $E<0$ and in terms of a parameter $0 \leq e < 1$

$$t = \sqrt{\frac{a^3}{GM}}\,(\chi - e\sin\chi) = a\sqrt{\frac{\mu}{2E}}\,(\chi - e\sin\chi)$$

$$\rho = a\,(1 - e\cos\chi) \qquad \text{with } a = \frac{\alpha}{2|E|} = \frac{GM\mu}{2|E|}$$

$$\varphi = 2\arctan\left[\sqrt{\frac{1+e}{1-e}}\,\tan\frac{\chi}{2}\right]$$

You can check that they satisfy $L = \alpha\sqrt{2\mu|E|(1-e^2)}$

$$\frac{d\rho}{dt} = \frac{d\rho}{d\chi}\frac{1}{\frac{dt}{d\chi}} = \pm\left(\frac{2}{\mu}\left(-|E| + \frac{GM\mu}{\rho}\right) - \frac{L^2}{\mu^2\rho^2}\right)^{1/2}$$

$$\frac{d\varphi}{dr} = \frac{d\varphi}{d\chi}\frac{1}{\frac{dt}{d\chi}} = \pm\left(\frac{2}{\mu}\left(-|E| + \frac{GM\mu}{\rho}\right) - \frac{L^2}{\mu^2\rho^2}\right)^{-1/2}\frac{L}{\mu\rho^2}$$

or equivalently $E = \frac{1}{2}\mu\dot{\rho}^2 + \frac{L^2}{2\mu\rho^2} - \frac{GM\mu}{\rho}$, $L = \mu\rho^2\dot{\varphi}$

are constant.

This curve is an ellipses of eccentricity e and a is the further point from the centre (apo-astron).

For $E = 0$ we have a parabolic motion, while when $E > 0$ the trajectory is hyperbolic. The case $e = 0$, $E < 0$ describe a circular motion. We'll focus on this case.

Plan: (4) GW from non-relativistic binaries

We can now evaluate I_{ij} on pag. 4

$$I_{ij}(0) = \int d^3x' \, T^{00} x'_i x'_j$$

for the case where T^{00} is the stress tensor of a particle of mass $\mu = \frac{m_1 m_2}{m_1 + m_2}$ which describes a binary whose constituents have masses m_1 and m_2. We have

$$T_{ab}(y) = m \, U_a \, U_b$$

$U_a = \frac{d y^a}{d\tau} = \gamma(1, v_i)$ is the 4-velocity of the matter in y^a

It's the mass density of the matter: for a particle localised in x^a: $m = \mu \, \delta^3(y^a - x^a)$

(where $\delta(\ldots)$ is the Dirac delta $\int dx \, f(x) \, \delta(x) = f(0)$).

Since we are approximating the binary with a point-line object following the trajectory $x^a(t)$ the integrals in I_{ij} can be performed trivially by using the δ

$$I_{ij} = \mu \, x_i \, x_j$$

At leading order as $\frac{v}{c} \ll 1$ we can use the trajectories derived above with $x_1 = \rho\cos\phi$, $x_2 = \rho\sin\phi$.

Thus by implementing the symmetric-traceless property (see end of pag. 7) we get

$$\boxed{I_{\langle ij\rangle} = \mu\left\{\begin{pmatrix} x_1^2 & x_1x_2 & 0 \\ x_1x_2 & x_2^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{1}{3}(x_1^2+x_2^2)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}}$$

where I used $x_3=0$ (the motion takes place in the equatorial plane). Thus the physical polarizations read

$$\bar{h}_{ij} = \frac{2G}{r} P_{ijkl}\underbrace{\left(\partial_0^2 I_{\langle ij\rangle}\right)}_{U_{\langle ij\rangle}}$$

The result is somewhat complicated for elliptic orbits: one need to invert the relation between t and χ and the explicit expressions for $\cos\phi$, $\sin\phi$ are involved. Things simplify in the case of circular orbits $(e=0)$ ... which happens to be phenomenologically interesting as most of the events detected fall in this category!

Then for $e=0$ we have $\rho = a$ and

$$\varphi = 2\arctan\left[\sqrt{\frac{1+e}{1-e}}\tan\frac{\chi}{2}\right] \Rightarrow \frac{\varphi}{2} = \frac{\chi}{2}$$

$$t = a\sqrt{\frac{\mu}{2|E|}}\left(\chi - e\sin\chi\right) \Rightarrow t = a\sqrt{\frac{\mu}{2|E|}}\,\chi = a\sqrt{\frac{\mu}{2E}}\,\varphi$$

$$x_1 = a\cos\left(\overbrace{\sqrt{\frac{2|E|}{\mu}}\frac{t}{a}}^{\Omega_{orb}}\right) \;;\; x_2 = a\sin\left(\sqrt{\frac{2|E|}{\mu}}\frac{t}{a}\right) \Rightarrow$$

$\Omega_{orb} = \sqrt{\frac{2|E|}{\mu}}\frac{1}{a}$ is the orbital frequency

$$x_1^2 = a^2\frac{1}{2}\cos\left(\Omega_{GW}t\right) + \dots \;;\; x_2^2 = a^2\frac{1}{2}\left(-\cos\Omega_{GW}t + \dots\right)$$

$$x_1x_2 = a^2\frac{1}{2}\sin\left(\Omega_{GW}t + \dots\right) \text{ where } \Omega_{GW} = 2\sqrt{\frac{2|E|}{\mu}}\frac{1}{a} = 2\Omega_{orb}$$

and the dots stand for time-independent terms which are killed by the time derivatives $U \sim \partial_0^2 I$ (I am using trig-identities such as $\cos 2x = 2\cos^2 x - 1$ etc.).

In summary we have

retarded time $u = t - r$

$$U_{\langle ij\rangle} \simeq \mu\frac{a^2}{2}\left\{\partial_0^2\left[\begin{pmatrix}\cos(\Omega_{GW}u) & \sin(\Omega_{GW}u) & 0\\ \sin(\Omega_{GW}u) & -\cos(\Omega_{GW}u) & 0\\ 0 & 0 & 0\end{pmatrix} + \dots\right]\right\}_0$$

and $\bar{h}_{ij} \simeq \frac{2G}{r}P_{ijkl}\,U_{\langle kl\rangle}$

Comments on this result

- it describes a monochromatic wave of frequency

$$\Omega_{GW} = 2\sqrt{\frac{2|E|}{\mu}\frac{1}{a}} = 2\sqrt{\frac{GM\mu}{a\mu}\frac{1}{a}} = 2\,\frac{\sqrt{GM}}{a^{3/2}}$$

using the relation between a and E on pag. 11

- For our circular motion we have

$$\frac{1}{2}\mu v^2 - \frac{GM\mu}{a} = E = -\frac{GM\mu}{2a} \Rightarrow \frac{1}{2}\mu v^2 = \frac{GM\mu}{2a} = |E|$$

- The orbital frequency Ω_{orb} is a uniform circular motion

is $\Omega_{orb} = \frac{v}{a} = \frac{\sqrt{GM}}{a^{3/2}} \Rightarrow \Omega_{GW} = 2\Omega_{orb}$ the frequency of the wave is twice that of the orbital motion

- This relation betwee orbital frequency and the radius of the orbit is Kepler law (!)

... however this linear analysis is missing a crucial ingredient that we'll cover next time