MTH5113 (2022/23): Problem Sheet 11

These problems are for practice only. *You do not need to submit them.*

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(1) *(Boring computations—but you should know how to do them)* Consider the following vector fields $\mathbf{F}$, $\mathbf{G}$, $\mathbf{H}$ on $\mathbb{R}^3$, where:

$$\begin{aligned}
\mathbf{F}(x,y,z) &= (x,\ y,\ z)_{(x,y,z)},\\
\mathbf{G}(x,y,z) &= (x^2,\ -2xy,\ 3xz)_{(x,y,z)},\\
\mathbf{H}(x,y,z) &= (x^2+y^2+z^2,\ x^4-y^2z^2,\ xyz)_{(x,y,z)}.
\end{aligned}$$

(a) Compute the divergence of $\mathbf{F}$, $\mathbf{G}$, and $\mathbf{H}$ at each point $(x,y,z) \in \mathbb{R}^3$.

(b) Compute the curl of $\mathbf{F}$, $\mathbf{G}$, and $\mathbf{H}$ at each point $(x,y,z) \in \mathbb{R}^3$.

(2) *(Fun with Green's theorem)* Let C denote the circle

$$C = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 9\},$$

and let us assign to C the anticlockwise orientation.

(a) Let $\mathbf{F}$ be the vector field on $\mathbb{R}^2$ defined by

$$\mathbf{F}(x,y) = (x-y,\ x+y)_{(x,y)}.$$

Compute the curve integral of $\mathbf{F}$ over C *directly*.

(b) Compute the curve integral from part (a) *using Green's theorem*. Check that your answer here matches the answer you obtained in part (a).

(c) Let $\mathbf{G}$ be the vector field on $\mathbb{R}^2$ defined by

$$\mathbf{G}(x,y) = (x^{9999999999999}e^x + y,\ x + y^{33333333333333}e^{2y})_{(x,y)}.$$

Compute (using your favourite method) the curve integral of $\mathbf{G}$ over C.

(3) *(Fun with Stokes' theorem)* Let S denote the upper half-sphere,

$$S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1,\ z > 0\}.$$

In addition, let C denote the boundary of S, i.e. the circle

$$C = \{(x, y, 0) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\},$$

and assign to C the (anticlockwise) orientation generated by the parametrisation

$$\gamma : \mathbb{R} \to \mathbb{R}^3, \qquad \gamma(t) = (\cos t,\ \sin t,\ 0).$$

(a) Let $\mathbf{F}$ be the vector field on $\mathbb{R}^3$ given by

$$\mathbf{F}(x, y, z) = (x,\ y,\ z)_{(x,y,z)}.$$

Compute directly the curve integral of $\mathbf{F}$ over C.

(b) Evaluate the integral from part (a) by applying *Stokes' theorem* and computing instead an appropriate integral over S. Make sure that you obtain the same answer as in (a).

(c) Let $\mathbf{G}$ and $\mathbf{H}$ be smooth vector fields on $\mathbb{R}^3$, and assume that $\mathbf{G}(\mathbf{p}) = \mathbf{H}(\mathbf{p})$ for every $\mathbf{p} \in C$. Using Stokes' theorem, show that

$$\iint_S (\nabla \times \mathbf{G}) \cdot d\mathbf{A} = \iint_S (\nabla \times \mathbf{H}) \cdot d\mathbf{A}.$$

(Here, S can be assigned either of its orientations.)

(4) *(Fun with the divergence theorem)* Let $\mathbb{S}^2$ denote the unit sphere centred at the origin,

$$\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\},$$

and assign to $\mathbb{S}^2$ the outward-facing orientation.

(a) Let $\mathbf{F}$ be the vector field on $\mathbb{R}^3$ given by

$$\mathbf{F}(x, y, z) = (x,\ y,\ z)_{(x,y,z)}.$$

Compute directly the surface integral of $\mathbf{F}$ over $\mathbb{S}^2$.

(b) Evaluate the integral from part (a) by applying the *divergence theorem* and computing an appropriate triple integral. Make sure that you obtain the same answer as in (a).

(c) Let $\mathbf{L}$ be the vector field on $\mathbb{R}^3$ given by

$$\mathbf{L}(x, y, z) = \left(y^{543} e^{y^2+z^4} z^{5234},\ e^{x^{562}z^{27}-x^{12}z^{43}} (x + ze^x)^{127},\ 1 + x^{10}y + 24y^{17}e^{42y^3}\right)_{(x,y,z)}.$$

Evaluate the surface integral of $\mathbf{L}$ over $\mathbb{S}^2$.

(5) *(Setting boundaries I)*

(a) Describe the boundaries of the following subsets of $\mathbb{R}^2$, as one or more curves in $\mathbb{R}^2$:

(i) $D_1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + 4y^2 < 4\}$.

(ii) $D_2 = (0, 1) \times (0, 2)$.

(iii) $D_3 = \{(x, y) \in \mathbb{R}^2 \mid x < 1,\ y > 0,\ y < x\}$.

(b) Describe the boundaries of the following surfaces in $\mathbb{R}^3$, as one or more curves in $\mathbb{R}^3$:

(i) $S_1 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1,\ x > 0\}$.

(ii) $S_2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1,\ -1 < z < 1\}$.

(iii) $S_3 = \{(x, y, 0) \in \mathbb{R}^3 \mid x \in (0, 1),\ y \in (0, 2)\}$.

(c) Describe the boundaries of the following regions in $\mathbb{R}^3$, as one or more surfaces in $\mathbb{R}^3$:

(i) $V_1 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 < 9\}$.

(ii) $V_2 = (0, 1) \times (0, 1) \times (0, 1)$.

(iii) $V_3 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 < 1,\ -1 < z < 1\}$.

(6) *(Setting boundaries II)*

(a) Suppose you apply Green's theorem to each of the regions D_i in Question (5a). Describe the resulting orientation(s) on the boundary of each D_i that you would obtain, according to the statement of Green's theorem.

(b) Suppose you apply Stokes' theorem to each of the surfaces S_i in Question (5b), where:

- For S_1, we use the orientation facing the positive x-direction.
- For S_2, we use the outward-facing orientation.
- For S_3, we use the orientation facing the positive z-direction.

Describe the resulting orientation(s) on the boundary of each S_i that you would obtain, according to the statement of Stokes' theorem.

(c) Suppose you apply the divergence theorem to each of the regions V_i in Question (5c). Describe the resulting orientation(s) on the boundary of each V_i that you would obtain, according to the statement of the divergence theorem.

(7) *(Second derivative identities)*

(a) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function. Show that the curl of the gradient of f vanishes everywhere, that is, show that for any $\mathbf{p} \in \mathbb{R}^3$,

$$[\nabla \times (\nabla f)](\mathbf{p}) = (0, 0, 0)_{\mathbf{p}}.$$

(b) Let $\mathbf{F}$ be a smooth vector field on $\mathbb{R}^3$. Show that the divergence of the curl of $\mathbf{F}$ vanishes everywhere, that is, show that for any $\mathbf{p} \in \mathbb{R}^3$,

$$[\nabla \cdot (\nabla \times \mathbf{F})](\mathbf{p}) = 0.$$

(8) *(Connections to Complex Variables)* *(Not examinable)* Consider the plane $\mathbb{R}^2$, or equivalently, the complex plane $\mathbb{C}$. Let C denote the unit circle centred at the origin,

$$C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \simeq \{z \in \mathbb{C} \mid |z| = 1\},$$

and assign to C the anticlockwise parametrisation. Furthermore, let $f : \mathbb{C} \to \mathbb{C}$ be a complex-analytic function, and write f in terms of its components as

$$f(z) = u(z) + i \cdot v(z), \qquad z \in \mathbb{C}.$$

(a) Express the real and imaginary parts of the contour integral of f over C,

$$\mathrm{Re} \int_C f(z)\, dz, \qquad \mathrm{Im} \int_C f(z)\, dz,$$

as curve integrals of appropriate vector fields over C.

(b) Use Green's theorem to prove Cauchy's theorem on C:

$$\int_C f(z)\, dz = 0.$$

(Hint: Recall that u and v satisfy the Cauchy-Riemann equations.)

(9) *(Green's theorem fail)* Let C be as in Question (8), and let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x, y) = \left(-\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2}\right)_{(x,y)}, \qquad (x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}.$$

(a) Show that for any $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$,

$$\partial_x \left(\frac{x}{x^2 + y^2}\right) - \partial_y \left(-\frac{y}{x^2 + y^2}\right) = 0.$$

(b) On the other hand, show that the integral of $\mathbf{F}$ over C is *not* zero. Why does this not contradict the statement of Green's theorem? *(More specifically, why do Green's theorem and part (a) not imply that the integral of $\mathbf{F}$ over C is zero?)*

(c) *(Not examinable)* For those taking *Complex Variables*, can you relate what you saw in parts (a) and (b) to some contour integrals that you have seen?