MTH5113 (2023/24): Problem Sheet 10

All coursework should be submitted individually.

- Problems marked "[**Marked**]" should be submitted and will be marked.

Please submit the completed problem on QMPlus:

- At the portal for **Coursework Submission 2**.

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(1) *(Warm-up)* Consider the following constrained optimisation problem:

- *Maximise the function* $f(x, y) = x$*, subject to the constraint* $x^2 + y^2 = 1$*.*

(a) Give the solution to the above problem *without doing any calculations. (The answer should be obvious from inspection alone; draw a picture if you are not sure.)*

(b) Solve the above problem using the method of Lagrange multipliers. Verify that your solution matches what you deduced in part (a).

(2) *(Warm-up)* Consider the following constrained optimisation problem:

- *Minimise the function* $f(x, y, z) = z$*, subject to the constraint* $x^2 + y^2 + z^2 = 1$*.*

(a) Give the solution to the above problem *without doing any calculations. (The answer should be obvious from inspection alone; draw a picture if you are not sure.)*

(b) Solve the above problem using the method of Lagrange multipliers. Verify that your solution matches what you deduced in part (a).

(3) [**Marked**] Solve the following problem using the method of Lagrange multipliers:

- *Find the maximum and minimum values of* x^2y^2*, subject to the constraint*

$$\left(\left(x - \frac{3}{2} \right)^2 + y^2 \right) \left(\left(x + \frac{3}{2} \right)^2 + y^2 \right) = 9 \ .$$

At which points are the maximum and minimum values achieved?

(4) *(Differential Geometry and Game Theory)* Let x^2 be the number of hours of *MTH5113* lectures and tutorials you attend, and let y^2 be the number of hours of *MTH5113* lectures and tutorials you skip. As you know, the *constraint* is that there are only 43 total hours of lectures and tutorials in *MTH5113*. Now, suppose that the "effectiveness" of your learning in *MTH5113*, as a function of the above hours spent, is modelled by the relation

$$E = 100x^2 + y^2.$$

(Here, a higher value of E *means better learning!)* Your objective here is to *optimise* the "effectiveness" of your learning experience in *MTH5113*!

(a) Express the above objective as a constrained optimisation problem.

(b) Use the method of Lagrange multipliers to solve the problem in part (a).

(c) Given your answer in (b), what optimal strategy should you adopt in order to have the most effective learning experience in *MTH5113*? :)

(5) [**Tutorial**] Use the method of Lagrange multipliers to solve the following:

(a) Find the minimum and maximum of $4x^2 - y^2$, subject to the constraint $x^2 + 4y^2 = 4$.

(b) Find the unit vectors $(x, y, z) \in \mathbb{R}^3$ that maximise and minimise the dot product,

$$(6, -3, 2) \cdot (x, y, z).$$

(6) *(Conservative and liberal vector fields)*

(a) Let f be the real-valued function

$$f : \mathbb{R}^3 \to \mathbb{R}, \qquad f(x, y, z) = x^4 y^2 z.$$

Compute the integral of the vector field ∇f over the curve

$$C = \{(t, t^2, t^3) \in \mathbb{R}^2 \mid t \in (0, 1)\},$$

where C is given the *rightward* (i.e. in the direction of increasing x-value) *orientation.*

(b) Let $g : \mathbb{R}^2 \to \mathbb{R}$ be given by the formula

$$g(x, y) = x^{17} e^{y + x^2 y^5 \cos x^7} + y^4 + e^{x^2 + y^2 + x^{42} + y^{1776} e^{yx^2}}.$$

Find the integral of the vector field ∇g over the unit circle

$$\mathcal{C} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\},$$

where $\mathcal{C}$ is given the *anticlockwise orientation.*

(c) Integrate the vector field

$$\mathbf{H}(x, y) = (-y, x), \qquad (x, y) \in \mathbb{R}^2,$$

over the unit circle $\mathcal{C}$ from part (b), where $\mathcal{C}$ again has the anticlockwise orientation.

(d) From your answer in part (c), conclude that the vector field $\mathbf{H}$ cannot be the gradient of any real-valued function $h : \mathbb{R}^2 \to \mathbb{R}$.

(7) *(Lagrangian Formulation of Multipliers)* Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be smooth functions, and fix $c \in \mathbb{R}$. Show that the following conditions are equivalent:

(i) $(x, y) \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$ satisfy the following system of equations:

$$\nabla f(x, y) = \lambda \cdot \nabla g(x, y), \qquad g(x, y) = c.$$

(ii) $(x, y, \lambda) \in \mathbb{R}^3$ satisfies the equation

$$\nabla \mathcal{L}(x, y, \lambda) = (0,\ 0,\ 0)_{(x,y,\lambda)},$$

where the function $\mathcal{L}$, called the *Lagrangian*, is defined by

$$\mathcal{L} : \mathbb{R}^3 \to \mathbb{R}, \qquad \mathcal{L}(u, v, w) = f(u, v) - w[g(u, v) - c].$$

(Thus, the method of Lagrange multipliers could also be formulated in terms of $\mathcal{L}$ in (ii)—if the maximum or minimum of f, subject to the constraint g, is achieved at (x, y), then there is some $\lambda \in \mathbb{R}$ such that (x, y, λ) is a critical point of the Lagrangian $\mathcal{L}$.)

(8) *(Multiple Constraints)* Assume the following formal setting:

- Let $U \subseteq \mathbb{R}^3$ be open and connected.
- Let $f : U \to \mathbb{R}$, $g : U \to \mathbb{R}$, $h : U \to \mathbb{R}$ be smooth functions.

- Suppose $\nabla g(\mathbf{p}) \times \nabla h(\mathbf{p})$ is nonvanishing at every $\mathbf{p} \in U$.

Under the above assumptions, the following result holds:

- **Theorem.** Suppose f achieves its maximum or minimum value on

$$C = \{(x, y, z) \in U \mid g(x, y, z) = c,\ h(x, y, z) = d\}$$

at a point $\mathbf{p} \in C$. Then, there exist $\lambda, \mu \in \mathbb{R}$ such that

$$\nabla f(\mathbf{p}) = \lambda \cdot \nabla g(\mathbf{p}) + \mu \cdot \nabla h(\mathbf{p}).$$

Using the preceding theorem:

(a) Devise a corresponding *method of Lagrange multipliers* for solving the following constraint optimisation problem: *maximise or minimise* $f(x, y, z)$, *subject to the simultaneous constraints* $g(x, y, z) = c$ *and* $h(x, y, z) = d$.

(b) *Using your strategy from part (a)*, find the maximum and minimum values of $x+y+z$, subject to the simultaneous constraints $x^2 + y^2 = 1$ and $x - z = 1$.

(>9000) *(Extra Exploration)* Put your geometry, calculus, and linear algebra knowledge to the test! Can you prove the theorem stated in Question (8)?

(Hint: The starting point is to observe that C *is a curve, by the result of Question (9) of Problem Sheet 4. From here, you have all the background you need to do this!)*

(Note: While I will not be posting the solution to this problem, I would be happy to chat with anyone who wishes to attempt it. Good luck!)