MTH 4104 Mock Exam Paper (2023-2024)

Shu SASAKI

This mock paper is purposely slightly harder (but only just) than the actual paper I've written this academic year.

Q1. Let $\mathcal{R}$ be a relation on the set of positive integers:

$$a\mathcal{R}b \Leftrightarrow \text{either } a \text{ divides } b\text{, or } b \text{ divides } a$$

Is this an equivalence relation? If so prove it. If not, explain exactly which axioms it fails to satisfy, by giving an explicit counter-example for each.

Q2. Solve the following set of equations in X and Y in $\mathbb{F}_{13}$:

$$\begin{aligned} X + 4Y &\equiv 17 \mod 13 \\ X - 2Y &\equiv 6 \mod 13 \end{aligned}$$

Q3. (1) Let G be the set of real numbers that are not equal to -1. Define a binary operation $*$ on G by

$$a * b = a + b + ab.$$

Prove that $(G, *)$ is a group.

(2)[Extra for Enthusiasts] Let S be a set consisting of four symbols $\{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$. Define a binary operation $*$ on S by the following table which describes (row) $*$ (column):

	$\clubsuit$	$\diamondsuit$	$\heartsuit$	$\spadesuit$
$\clubsuit$	$\clubsuit$	$\diamondsuit$	$\heartsuit$	$\spadesuit$
$\diamondsuit$	$\diamondsuit$	$\heartsuit$	$\spadesuit$	$\clubsuit$
$\heartsuit$	$\heartsuit$	$\spadesuit$	$\clubsuit$	$\diamondsuit$
$\spadesuit$	$\spadesuit$	$\clubsuit$	$\diamondsuit$	$\heartsuit$

Is $(S, *)$ a group? Justify your answer.

Q4. Let $(R, +, \times)$ be a ring and 0 denote the identity element with respect to addition $+$. Prove that $a0 = 0a = 0$ for every element a in R.

Q5. Let $\mathbb{H}$ be Hamilton's quaternions, i.e. the set of elements of the form

$$c1 + c(p)p + c(q)q + c(r)r \in \mathbb{R}1 + \mathbb{R}p + \mathbb{R}q + \mathbb{R}r$$

where the basis elements $1, p, q$ and r satisfy the multiplicative relations

- $1p = p1 = p$, $1q = q1 = q$, $1r = r1 = r$,
- $p^2 = -1, q^2 = -1, r^2 = -1$,
- $pq = r, qp = -r$,
- $qr = p, rq = -p$,

- $rp = q, qr = -q$,

together with natural addition and multiplication (prescribed by the relation).

(1) What is the multiplicative inverse of $p+q-r$? (2) Is $\mathbb{H}$ a field? If so, prove it. If not, explain why.

Q6. Find polynomials $f(X)$ and $g(X)$ in $\mathbb{F}_3[X]$ such that $(X^8+[2])f(X)+([2]X^6+[2])g(X) = \gcd(X^8+[2], [2]X^6+[2])$ in $\mathbb{F}_3[X]$.

Q7. Let σ be an element of S_{10} of the form

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 4 & 7 & 9 & 6 & 8 & 1 & 5 & 10 & 3 & 2 \end{pmatrix}$$

(1) Write σ in cycle notation . (2) Let τ be $(1)(2\,8\,6\,7)(3\,5\,4\,9)(10)$. Compute $\sigma \circ \tau^{-1}$ in cycle notatioon. (3) Determine the order of σ.