REVISION NOTES MATH5105 : DIFFERENTIAL AND INTEGRAL ANALYSIS

1. Revision

Let $\Omega \subseteq \mathbb{R}$ be a domain (i.e. an interval or all of $\mathbb{R}$).

Definition. Let $f : \Omega \to \mathbb{R}$.

The function f is continuous at $a \in \Omega$ if

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x \in \Omega \mid |x - a| < \delta \implies |f(x) - f(a)| < \varepsilon.$$

Theorem. *If $f : \Omega \to \mathbb{R}$ is continuous at $a \in \Omega$ and $f(a) \neq 0$ then $f(x) \neq 0$ in a neighbourhood of a that is*

$$\exists \delta > 0 \forall x \in D \mid |x - a| < \delta \implies f(x) \neq 0.$$

Proof. Since f is continuous at a and $b = f(a)$ so that

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x \in D \mid 0 < |x - a| < \delta \implies |f(x) - b| < \varepsilon.$$

We choose______ so that $|f(x) - b| < \frac{|b|}{2}$ so that by the reverse triangle inequality

$$\frac{|b|}{2} > |f(x) - b| \geq |b| - |f(x)|$$

or

$$|f(x)| > \frac{|b|}{2} > 0.$$

This shows

$$\exists \delta > 0 \forall x \in D \mid |x - a| < \delta \implies f(x) \neq 0.$$

$\square$

Theorem (Boundedness Principle)**.** *Let $f : [a, b] \to \mathbb{R}$ be a real-valued continuous function f on the interval $[a, b]$ then f ______________________________ That is if $m = \inf_{x \in [a,b]} f(x), M = \sup_{x \in [a,b]} f(x)$ then there exists $x_m, x_M \in [a, b]$ such that $f(x_m) = m$ and $f(x_M) = M$.*

Theorem (Intermediate Value Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be a real-valued continuous function f on the interval $[a, b]$. Then f__________________________________, that is for any c between $f(a)$ and $f(b)$ there exists an $x \in [a, b]$ such that $f(x) = c$.*

2. Differentiation

Definition. Let $x_0 \in (a,b), f : (a,b) \to \mathbb{R}$. The ________________ is defined as

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

If this limit exists then f is differentiable at x_0.

3. The Mean Value Theorem

Theorem (Mean Value Theorem)**.** *Suppose that $f : [a,b] \to \mathbb{R}$ is ________________ and differentiable on (a,b). Then ________________________*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

4. Exponential Function

Definition. A differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f'(x) = f(x) \quad \forall x \in \mathbb{R}$$

and _______ is called the exponential function.

Theorem. *The exponential function f defined above satisfies $f(x)f(-x) = 1$.*

Proof. Let $h(x) = f(x)f(-x)$. __________________________, we get

$$h'(x) = f'(x)f(-x) + f(x)f'(-x)(-1) = 0$$

and hence h is a constant. As $h(0) = f(0)f(0) = 1$ so $h(x) = 1$. $\square$

5. Inverse Functions

Theorem (Inverse Function Theorem)**.** *Let f be a _________________________ on an open interval I and let $J = f(I)$. If f is differentiable at $x_0 \in I$ and if $f'(x_0) \neq 0$ then f^{-1} is differentiable at $y_0 = f(x_0)$ and*

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$

Example. The function $f(x) = \sin(x)$ ____________________ and it has an inverse g restricted to this domain. The inverse g is usually denoted by $\sin^{-1}$ or arcsin. Note that $dom(g) = [-1, 1]$. For $y_0 = \sin x_0 \in (-1, 1)$ where $x_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$ then the inverse function formula shows

$$g'(y_0) = \frac{1}{\cos x_0}.$$

Since______________________________ we may write

$$g'(y_0) = \frac{1}{\sqrt{1 - y_0^2}}, \quad y_0 \in (-1, 1).$$

6. Higher Order Derivatives

Theorem. *Let $f, g : \Omega \to \mathbb{R}$ be differentiable for $|x - a| < \varepsilon$ and let $g'(x) \neq 0$ for$0 < |x - a| < \varepsilon$. If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ and if $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists then $\lim_{x\to a} \frac{f(x)}{g(x)}$ exists and*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Example. (1) We apply L'Hôpital's rule to

$$\lim_{x\to 0} \frac{\sqrt{1+2x} - \sqrt{1-x}}{x} = \lim_{x\to 0} \frac{1/\sqrt{1+2x} - 1/2\sqrt{1+x}}{1} = 1 - \frac{1}{2} = \frac{1}{2}.$$

(2) We apply L'Hôpital's rule twice

$$\lim_{x\to 0} \frac{\exp(x) - 1 - x}{x^2} = \lim_{x\to 0} \frac{\exp(x) - 1}{2x} = \lim_{x\to 0} \frac{\exp(x)}{2} = \frac{1}{2}.$$

7. Definition of the Riemann Integral

Definition. A partition P of $I = [a, b]$ is a set of points $\{x_0, x_1, \ldots, x_{n-1}, x_n\}$ such that

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

We denote the set of all partitions of I by $\mathcal{P}$. Let $I_i = [x_{i-1}, x_i], \triangle x_i = x_i - x_{i-1}$ for $i = 1, 2, \cdots, n$. A partition is call equidistant if $\triangle x_i = \frac{b-a}{n}$ is constant.

The partition P' is a refinement of P if $P' \supset P$.

We define the mesh size of a partition as

$$\sigma(P) = \max\{\triangle x_i \mid i = 1, 2 \cdots, n\}$$

If $P' \supseteq P$ is a refinement of P then $\sigma(P') \leq \sigma(P)$.

Definition. Let $f : [a, b] \to \mathbb{R}$ be bounded and $P = \{x_0, x_1, \cdots, x_n\}$ be a partition of $[a, b]$ We define the lower sum of f with respect to P as

$$L(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = \sum_{i=1}^{n} m_i \triangle x_i$$

where $m_i = \inf_{x \in [x_i, x_{i+1}]} f(x)$. The upper sum is defined as

$$U(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = \sum_{i=1}^{n} M_i \triangle x_i$$

where $M_i = \sup_{x \in [x_i, x_{i+1}]} f(x)$.

Definition (Uniform Continuity)**.** Suppose that $f : \Omega \to \mathbb{R}$ where I is an interval. We say that f is uniformly continuous on Ω if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon)$ such that____More compactly this means

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x, y \in \Omega \mid |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$$

Theorem. *Suppose that f is continuous on a closed, bounded interval $[a, b]$. Then f is uniformly continuous on $[a, b]$.*

8. Properties of the Riemann Integral

Theorem (Mean Value Theorem for Integrals)**.** *Let f* _____________________________________ *such that*

$$\int_a^b f(x)dx = f(c)(b - a).$$

9. The Fundamental Theorem of Calculus

Theorem (Fundamental Theorem of Calculus)**.** *Assume that $f : [a, b] \to \mathbb{R}$ is continuous. Define $F(x) = \int_a^x f(t)dt$* ___________________________

$$\int_a^b f(x)dx = F(b) - F(a).$$

10. Sequences and Series of Functions

Definition (Uniform Convergence)**.** We say f_n converges to f uniformly on Ω if for every $\varepsilon > 0 \exists N$ _______________________________ and all $x \in \Omega$.

Theorem. *Consider a sequence of continuous functions $f_n : [a,b] \to \mathbb{R}$ and suppose $\{f_n\}$ uniformly converges to f. Then f is also continuous.*

Theorem (Weierstrass M-test)**.** *Suppose that there exists a nonnegative sequence of ____________ and suppose that $f_n : \Omega \to \mathbb{R}$ satisfies $|f_n(x)| < M_n$ for all $x \in \Omega$. Then $\sum_{n=1}^{\infty} f_n$ converges uniformly.*

Example. Show that the series $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$ converges.

Proof. Note that ____________. Now the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges so by the Weierstrass test the above series converges uniformly. □

11. Power Series

Theorem. *For the power series $\sum a_n x^n$ let*

$$\beta = \limsup_{n \to \infty} |a_n|^{1/n} \quad \& \quad R = \frac{1}{\beta}.$$

If $\beta = 0$ we set $R = +\infty$ and if $\beta = +\infty$ we set $R = 0$. Then

(1) *the power series converges for $|x| < R$,*
(2) *the power series diverse for $|x > R$.*

Example. Consider the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2}.$$

Once again $\beta = 1$ and $R = 1$ and the series converges at $x = 1$ and $x = -1$ so the series has interval of convergence $[-1, 1]$.

Example. Recall that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Differentiating term by term we obtain

$$\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}, \quad |x| < 1.$$

Similarly integrating term by term

$$\sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1} = \int_0^x \frac{1}{1-t} dt = -\log(1-x)$$

or

$$\log(1-x) = -\sum_{n=1}^{\infty} \frac{1}{n} x^n, \quad |x| < 1.$$

Replacing x by $-x$ we get

$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}.$$