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Game theory

- Decision making when agents/players interact.
- Assume agents behave rationally.
- Many applications in Economics

Game Theory

Example 10.1. Suppose that Rosemary and Colin each have 2 cards, labelled with a 1 and a 2. Each selects a card and then both reveal their selected cards. If the sum s of the numbers on their cards is even, then Rosemary wins and Colin must pay her this s. Otherwise, Colin wins and Rosemary must pay him s.

Can represent this information in a payoff matrix.

Colin (player 2)

Rosemary (Player 1)	1	2
1	$(2,-2)$	$(-3,3)$
2	$(-3,3)$	$(4,-4)$

Example 12.1. Suppose that Rosemary and Colin are working on a joint project. Each of them can choose to "work hard" or "goof off." Both of them must work hard together to receive a high mark for the project. Both have utility 3 for receiving a high mark utility 1 for goofing off (regardless of what mark they receive) and utility 0 for working hard but not receiving a high mark. Give the payoff matrix for this game.

Colin (player 2

Rosemary (player 1)	work hard (w)	goof off (g)
(w) work hard	$(3,3)$	$(0,1)$
(g) goof off	$(1,0)$	$(1,1)$

Rosemary's set of strategies $A_1 = \{\text{work hard, goof off}\}$

Colin's set of strategies $A_2 = \{\text{work hard, goof off}\}$.

e.g. $u_1(w,g) = 0$

$u_2(g,g) = 1$

<u>Defn</u> A 2-player strategic game is a game with two players

player 1 / row player / Rosemary
player 2 / column player / Colin

The game is specified by

- A set of <u>strategies</u> A_1 for player 1 (i.e. choices)
  a set of strategies A_2 for player 2

- player 1's <u>payoff function</u> $u_1 : A_1 \times A_2 \to \mathbb{R}$
  player 2's payoff function $u_2 : A_1 \times A_2 \to \mathbb{R}$

i.e. if player 1 plays strategy $a_1 \in A_1$
player 2 plays strategy $a_2 \in A_2$

then payoff to player 1 is $u_1(a_1, a_2)$
- - - - player 2 is $u_2(a_1, a_2)$

<u>Remarks</u>

- A pair of strategies (a_1, a_2) with $a_1 \in A_1$ and $a_2 \in A_2$ is called an <u>outcome</u>

- "payoff" is numerical value of happiness of a player from a particular outcome.

- So payoff to any player depends on both on their choice of strategy and the other player's choice.

A <u>payoff matrix</u> for a 2-player game is a matrix

- rows are labelled by player 1's (Rosemary) strategies
  columns . . . . . player 2's (Colin) strategies
- For a strategy a_1 (resp. a_2) of player 1 (resp. player 2) the (a_1, a_2) entry of the matrix consists of

  Rosemary's payoff $u_1(a_1, a_2)$ followed by
  Colin's payoff $u_2(a_1, a_2)$.

<u>Assumption</u> (Principle of rational choice)
In any situation, a player will seek to maximise their payoff.

We mostly focus on zero-sum games.

<u>Defn</u> A 2-player strategic game is called a <u>zero-sum game</u> if for each outcome $(a_1, a_2) \in A_1 \times A_2$, we have $u_1(a_1, a_2) = -u_2(a_1, a_2)$
(i.e. payoffs of the two players sum to zero)

We simplify payoff matrix by only writing payoff to the row player (Rosemary)

zero sum game

Colin (player 2)

Rosemary (Player 1)	1	2
1	(2, -2)	(-3, 3)
2	(-3, 3)	(4, -4)

Simplified payoff matrix

	1	2
1	2	-3
2	-3	4

One more example

Example 10.3. Rosemary and Colin each have a £1 and a £2 coin. They each select one of them and hold it in their hand, then Colin calls out "even" or "odd" and they reveal their coins. Let s be the sum of the values of the coins. If Colin correctly guessed whether s was even or odd, he wins both coins. Otherwise, Rosemary wins both coins.

What are the strategies for each player?
Write down payoff matrix?
Is this a zero-sum game? Yes

Colin

Rosemary	(1, odd)	(2, odd)	(1, even)	(2, even)
1	(1, −1)	(−1, 1)	(−1, 1)	(2, −2)
2	(−2, 2)	(2, −2)	(1, −1)	(−2, 2)

Colin

Rosemary	(1, odd)	(2, odd)	(1, even)	(2, even)
1	1	−1	−1	2
2	−2	2	1	−2

In simplified payoff matrix

- Rosemary wants outcomes with high value
- Colin wants outcomes with low value.

Colin

Rosemary	c_1	c_2	
r_1	100	-50	security of r_1 = -50
r_2	8	20	security of r_2 = 8 (best)
	security of c_1 = 100	security of c_2 = 20 (best)	

(A is simplified payoff matrix)

<u>Defn</u> Suppose we have a 2-player zero-sum game where $R = \{r_1, \dots, r_k\}$ is the set of Rosemary's strategies
$C = \{c_1, \dots, c_l\}$ is the set of Colin's strategies

Security level of r_p = min entry in p^{th} row of A

$= \min_j a_{pj}$ (worst payoff for Rosemary if she plays r_p)

Security level of c_q = max entry in q^{th} column of A

$= \max_j a_{jq}$ (tells us worst payoff for Colin if he plays c_q with a minus sign)

Best security level for Rosemary is the highest of her security levels

Best security level for Colin is the lowest of his security levels.

Colin

Rosemary	c_1	c_2
r_1	100	-50
r_2	8	20

In example, playing accoring to security levels is "unstable": colin has an incentive to change his strategy.

In fact every outcome is "unstable"

This motivates idea of Nash equilibrium.

Informally, a pure Nash equilibrium is a pair of strategies (r_i, c_j) (i.e. an outcome) where neither player has an incentive to change their strategy unilaterally.

Defn Pure Nash equilibrium in zero-sum games

Consider a zero-sum game with payoff matrix $A = a_{ij}$ where $R = \{r_1, \dots, r_k\}$ is set of Rosemary's strategies and $C = \{c_1, \dots, c_\ell\}$ is set of Colin's strategies

A pair of strategies (r_i, c_j) is a Nash equilibrium if

$$a_{ij} \geq a_{i'j} \quad \text{for all } i' = 1, \dots, k$$

$$\text{and} \quad a_{ij} \leq a_{ij'} \quad \text{for all } j' = 1, \dots, \ell$$

i.e. a_{ij} is the largest entry in its column and smallest entry in its row

(so Rosemary cannot increase her payoff by choosing a different strategy from r_i assuming Colin stays at c_j

Colin cannot increase his payoff by choosing a different strategy from c_j assuming Rosemary stays at r_i.

Defn Pure Nash equilibrium in zero-sum games

Consider a zero-sum game with pay-off matrix $A = a_{ij}$ where $R = \{r_1, \dots, r_k\}$ is set of Roseman's strategies and $C = \{c_1, \dots, c_\ell\}$ is set of Colin's strategies

A pair of strategies (r_i, c_j) is a Nash equilibrium if

$$a_{ij} \geq a_{i'j} \quad \text{for all } i' = 1, \dots, k$$

$$\text{and} \quad a_{ij} \leq a_{ij'} \quad \text{for all } j' = 1, \dots, \ell$$

i.e. a_{ij} is the largest entry in its column and smallest entry in its row

(so Rosemary cannot increase her payoff by choosing a different strategy from r_i assuming Colin stays at c_j

Colin cannot increase his payoff by choosing a different strategy from c_j assuming Rosemary stays at r_i.

Example 10.4. Suppose we seek a pair of strategies (r_i, c_j) that form a Nash equilibrium for the game with the following payoff matrix:

	2	7	2	12	11	
	c_1	c_2	c_3	c_4	c_5	
r_1	2	−3	−3	12	−5	−5
r_2	2	7	2	9	11	2
r_3	−1	4	0	1	0	−1
r_4	−3	5	1	2	−3	−3

Can you find a Nash equilibrium?
How many can you find?

Thm Suppose we a 2-player zero-sum game.

Let u_r^* be best (highest) security level for row player
u_c^* be best (lowest) security level for column player

The game a Nash equilibrium if and only if

$$u_r^* = u_c^* .$$

Pf Let $A = a_{ij}$ be (simplified) payoff matrix.

For any strategy r_p for row player
and any strategy c_q for column player

$$\text{security level of } r_p \leq a_{pq} \leq \text{security level of } c_q \qquad (1)$$

$$\begin{array}{c|c} & c_q \\ \hline r_p & a_{pq} \end{array}$$

Suppose r_i has best (highest) security level for row player
c_j has best (lowest) security level for column player

$$u_r^* = \text{security level of } r_i \leq a_{ij} \leq \text{security level of } c_j = u_c^* \qquad (2)$$

from (1)

So $u_r^* \leq u_c^*$

Suppose $u_r^* = u_c^*$

Then from (2), we know

security level of r_i $= a_{ij} =$ security level of c_j

So a_{ij} is smallest entry in its row and largest entry in its column — using defn of security level

So (r_i, c_j) is a Nash equilibrium

For converse, suppose (r_p, c_q) is a Nash equilibrium

Then a_{pq} is smallest entry in its row and largest entry in its column — defn of Nash equilibrium

i.e. $a_{pq} =$ security level of $r_p \leq u_r^*$

$a_{pq} =$ security level of $c_q \geq u_c^*$

So $u_c^* \leq u_r^*$

But know $u_r^* \geq u_c^*$ by (2)

So $u_r^* = u_c^*$ $\square$

Example matching pennies

> **Example 10.5.** Rosemary and Colin each have a 1p coin. Simultaneously, they place their coins on the table with either heads or tails showing. If the coins match, Rosemary wins £1 from Colin. Otherwise, Colin wins £1 from Rosemary.

Zero-sum game with payoff matrix

$$
\begin{array}{cc|cc c}
 & & \multicolumn{2}{c}{\text{Colin}} & \\
 & & h & t & \\
\hline
\text{Rosemary} & h & 1 & -1 & -1 \\
 & t & -1 & 1 & -1 \\
 & & 1 & 1 &
\end{array}
$$

Has no pure Nash equilibrium (check using defn of using security levels with previous thm)

If Rosemary plays any strategy (h/t) consistently then eventually Colin plays opposite strategy (h/t) and win.

Rosemary should pick her strategy randomly to prevent this!

Defn (Mixed strategy)

Let $S = \{s_1, \dots, s_k\}$ be the set of strategies for a player. A mixed strategy is a vector $\underline{x} = (x_1, x_2, \dots, x_k) \in \mathbb{R}^k$ satisfying $x_1, x_2, \dots, x_k \geq 0$ and $x_1 + x_2 + \dots + x_k = 1$

Think of x_i as the probability of playing s_i.

We write $\Delta(S)$ for set of all mixed strategies (infinitely many)

Each $s_i \in S$ is called a pure strategy

		Colin	
		h	t
Rosemary	h	1	-1
	t	-1	1

$R = \{h, t\}$ is set of Rosemary's pure strategies

$C = \{h, t\}$ is set of Colin's pure strategies

e.g. $\underline{r} = \left(\frac{1}{3}, \frac{2}{3}\right)$ is a mixed strategy for Rosemary

$\underline{c} = \left(\frac{1}{6}, \frac{5}{6}\right)$ is a mixed strategy for Colin.

What is expected payoff to Rosemary if they use these mixed strategies?

$$\text{Expected payoff to Rosemary} = \sum_{\text{outcomes}} \left(\mathbb{P}(\text{outcome}) \times \text{payoff to Rosemary from outcome}\right)$$

$$= \mathbb{P}(h,h) \times 1 + \mathbb{P}(h,t) \times (-1) + \mathbb{P}(t,h) \times (-1) + \mathbb{P}(t,t) \times 1$$

$$= \frac{1}{3} \times \frac{1}{6} \times 1 + \frac{1}{3} \times \frac{5}{6} \times (-1) + \frac{2}{3} \times \frac{1}{6} \times (-1) + \frac{2}{3} \times \frac{5}{6} \times 1$$

$$= \left(\frac{1}{3}, \frac{2}{3}\right) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1/6 \\ 5/6 \end{pmatrix}$$

$$= \underline{r}^T A \underline{c}$$ where A is the payoff matrix.

For a general 2-player zero-sum game with

$R = \{r_1, \dots, r_k\}$ set of Rosemary's strategies

$C = \{c_1, \dots, c_\ell\}$ set of Colin's strategies

$A = a_{ij}$ payoff matrix.

If Rosemary plays mixed strategy $\underline{x} \in \Delta(R)$

Colin plays mixed strategy $\underline{y} \in \Delta(C)$ then

expected payoff to Rosemary

$$= \sum_{(r_i, c_j)} \mathbb{P}\left(\text{outcome is } (r_i, c_j)\right) a_{ij} = \sum_{(r_i, c_j)} x_i y_j a_{ij}$$

$$= \underline{x}^T A \underline{y}$$

expected payoff to Colin $= -\underline{x}^T A \underline{y}$

Intuitively the security level of $\underline{x}$ is

$$\min_{\underline{y} \in \Delta(C)} \underline{x}^T A \underline{y}$$

i.e, least expected payoff to Rosemary if she plays $\underline{x}$

Next theorem shows that minimising over all $\underline{y} \in \Delta(C)$ is the same as minimising over Colin's pure strategies