Week 9 Interest ~~te~~ Rate Term Structure

Interest rate modelling is the most important topic

- Interest rate derivatives 80%

  swaps, Credit derivatives debt portfolios

Q: why is modelling interest rates more complicated than share prices?

$r(t)$ $S(t)$

A: current time ✓ ✓

$\boxed{\text{term of the investment}}$ ✓ ✗

≠ AMI FMI

- Stochastic
- Continuous time
- arbitrage-free

Two main types of interest rates models:

X ① Heath - Jarrow - Morton approach

forward rate

√ ② Short - rate models

- Ito process
- Short rate

Vasicek model

Cox - Ingersoll - Ross model (CIR)

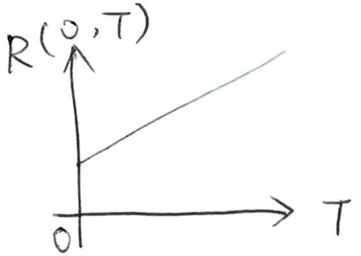
Hull - White model

Definition 14.1 Interet Rates Term Structure

How interest rates for different maturities are related

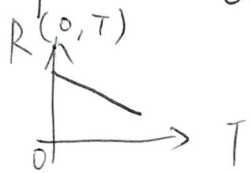
A function of interest rates on maturities, e.g. $R(0,T)$ or $r(0,T)$

Q: Typically, the yield curve increases with maturity. Why?

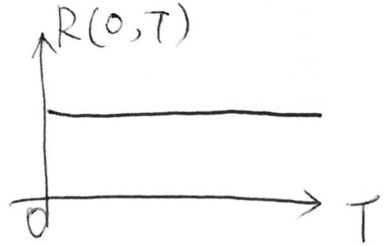


A: Reflecting uncertaity about far-future rates.

If the current rates are already unsually high,



Q: $R(0,T)$ is independent of T. what is the shape of the term structure?



Theories explaining the shape of the term structure:

① The expectation theory:

long-term rates determined by short-term rates

long-term bonds short-term bonds

② The market segmentation theory:

supply and demand

long-term rate: pension

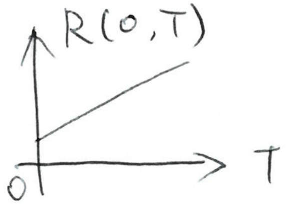
medium-term rate: business

short-term rate: market makers

③ The liquidity preference theory:

lenders : lend short term

borrowers : borrow long term



Slide 10 – 11 Th 14.1 No – arbitrage

Desirable characteristics of a term structure model

Q1: what are term structure models used for?

A: • bond traders: price inconsistencies → erase arbitrage opportunies
gain risk – free profit
price of interest rate derivatives

• investors : hedge

• Asset – liability modelling

logic : identify stakeholders

Equilibrium models: ~~to~~ economic theories, e.g. Vasicek, CIR
revert to the long-run average short rate
rarely replicate the reality

No-arbitrage models: eg Hull-White

Desirable characteristics:

① aribitrage free
② positive
③ mean-reverting
④ easy to calculate
⑤ reflect the ~~re~~ realistic dynamics, fit historical data
⑥ current market data
⑦ flexible cope with a range of derivative contracts.

14.2 The Vasicek Model (1977)

$r(t)$ $f(B_t)$ OUP

Definition 14.2

Vasicek model is the OUP:

$$dr(t) = -a(r(t)-\mu)\,dt + \sigma\,dW_t$$

where $a>0$, $\mu>0$, σ: $\sigma>0$ economically

σ any value mathematically.

Properties of Vasicek Model:

A review:

Th 13.4: If $r(t)$: $dr = -a(r-\mu)\,dt + \sigma\,dW_t$

Then $r(t) = b + (r(0)-\mu)e^{-at} + \sigma e^{-at}\int_0^t e^{as}\,dW_s$

Th 12.3: $\int_0^t f(s)\,dW_s \sim N\left(0, \int_0^t (f(s))^2\,ds\right)$

Propertiy 1: explicit solution

According to Th 13.4: If $r(t)$ is an OUP,

Then $$r(t) = \underbrace{(r_0 - \mu) e^{-at}}_{} + \underbrace{\mu}_{\triangle} + \underbrace{6 e^{-at} \int_0^t e^{as} \, dW_s}_{\text{Stochastic integral}}$$

Property 2: what happens if $t \to \infty$?

Th 12.3: $\int_0^t e^{as} \, dW_s \sim N\left(0, \int_0^t e^{2as} \, ds\right)$

$f(s)$

$E(r(t))$ and $Var(r(t))$

$$E(r(t)) = E\left[(r_0 - \mu) e^{-at} + \mu + 6 e^{-at} \int_0^t e^{as} \, dW_s\right]$$

$$= (r_0 - \mu) e^{-at} + \mu + 6 e^{-at} \underbrace{E\left[\int_0^t e^{as} \, dW_s\right]}_{=0}$$

$$= (r_0 - \mu) e^{-at} + \mu$$

$t \to \infty$ $\because a > 0$ $\therefore e^{-at} \to 0$ $\therefore E(r(t)) \to \mu$ as $t \to \infty$

Mean-reversion

$$Var(r(t)) = Var\left[(r_0 - \mu)e^{-at} + \mu + \sigma e^{-at} \int_0^t e^{as} dW_s \right]$$

$$= Var\left[\sigma e^{-at} \int_0^t e^{as} dW_s \right]$$

$$= \sigma^2 e^{-2at} \, Var\left[\int_0^t e^{as} dW_s \right]$$

$$= \sigma^2 e^{-2at} \int_0^t e^{2as} ds$$

$$Var(r(t)) = \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right)$$

$t \to \infty$ $\quad \because a > 0, \quad \therefore e^{-2at} \to 0 \quad \therefore Var(r(t)) \to \dfrac{\sigma^2}{2a}$

$Var(r(t))$ is independent of t, constant variance $\dfrac{\sigma^2}{2a}$

Property 3: distribution of $r(t)$

$$r(t) = (r_0 - \mu)e^{-at} + \mu + \sigma e^{-at} \underbrace{\int_0^t e^{as} dW_s}_{\text{Th 12.3} \quad N\left(0, \int_0^t e^{2as} ds\right)}$$

$r(t) \sim N\left((r_0-\mu)e^{-at}+\mu,\ \frac{\sigma^2}{2a}\left(1-e^{-2at}\right)\right)$

$t \to \infty \quad r(t) \sim N\left(\mu, \frac{\sigma^2}{2a}\right)$

For large values of t, the distribution of $r(t)$ does not depend on t.

Property 4: $r(t)$ can be negative $\leftarrow$ biggest disadvantage of Vasicek Model

However, the prob of $r(t)<0$ is small if σ is small

Q: $P(r(t)<0)$ for large value of t?

$\lim_{t\to\infty} p(r(t)<0) = ?$

A: $t \to \infty$, $r(t) \sim N\left(\mu, \frac{\sigma^2}{2a}\right)$ $\leftarrow$ From Property 3

$$\lim_{t\to\infty} p(r(t)<0) = P\Bigg(\underbrace{\boxed{\frac{r(t)-\mu}{\sqrt{\frac{\sigma^2}{2a}}}}}_{\sim N(0,1)} < \frac{0-\mu}{\sqrt{\frac{\sigma^2}{2a}}}\Bigg) \quad \text{standardisation}$$

$$= P\left(Z < \frac{-\mu\sqrt{2a}}{|\sigma|}\right) = \Phi\left(\frac{-\mu\sqrt{2a}}{|\sigma|}\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{-\mu\sqrt{2a}}{|\sigma|}} e^{-\frac{x^2}{2}}\,dx.$$

$|\sigma| \to 0$, $\frac{-\mu\sqrt{2a}}{|\sigma|} \to -\infty$, $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\infty} e^{-\frac{x^2}{2}} dx \to 0$

$$\lim_{t \to \infty} P(r(t) < 0) \to 0 \text{ if } |\sigma| \to 0$$

Property 5: mean reversion ← Advantage

$$dr(t) = -a\left(r(t) - \mu\right) dt + \sigma dW_t$$

If $r(t) - \mu > 0$ $\quad \underbrace{-a}_{<0} \underbrace{(r(t) - \mu)}_{>0} \underbrace{dt}_{>0} < 0$: negative force push $r(t)$ back to μ

$r(t) - \mu$ big deviation, ~~more~~ stronger negative force

If $r(t) - \mu < 0$ $\quad -a\underbrace{(r(t) - \mu)}_{<0} dt > 0$: positive force

The Cox-Ingersoll-Ross Model (1985)

Definition 14.3 CIR

$$dr(t) = -\alpha\,(r(t) - \mu)\,dt + \sigma \boxed{\sqrt{r(t)}}\,dW_t$$

$\alpha > 0,\ \mu > 0,\ \sigma > 0$

Property: if $\sigma^2 < 2\alpha\mu$, then $r(t) > 0$

$r(t)$ can be strictly positive if σ^2 is small enough.

The Hull-white Model (1990)

Definition 14.4 HW

$$dr(t) = -\alpha\,(r(t) - \boxed{\mu(t)})\,dt + \sigma\,dW_t$$

$\mu(t) > 0$ is a given deterministic function of t, $\alpha > 0$, $\sigma > 0$

$\mu \to \mu(t)$ more flexibility

Vasicek is a special example of HW $\mu(t) = \mu$