Assignment 4

Solutions

Assignment 4

Prove or disprove that $g \cdot g$ is RI $\Rightarrow$ g is RI

In the lecture notes we have g is RI $\Rightarrow$ g^2 is RI *

So we are asking whether the converse to is true? *

Lots of statements made which are just not true.

function bounded $\therefore$ RI ×

- bounded is needed to consider RI

function RI $\Rightarrow$ f continuous? ×

f not continuous $\Rightarrow$ f not RI ? ×

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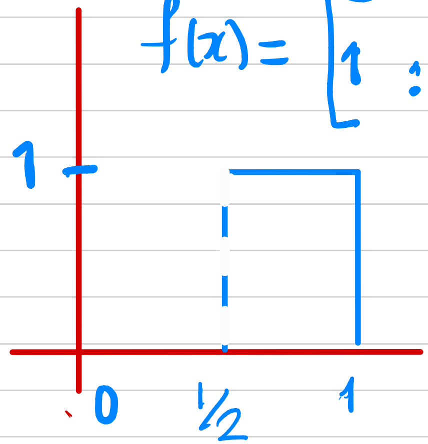
g^2 is RI $\Rightarrow$ g is RI ?

TRUE or FALSE

counterexample let $P_n = \{0, \frac{1}{2} - \frac{1}{n}, \frac{1}{2}, 1\}$

Show $U(f,P_n) - L(f,P_n) = \frac{1}{n}$

$$f(x) = \begin{cases} 0 : & 0 \le x < \frac{1}{2} \\ 1 : & \frac{1}{2} \le x < 1 \end{cases}$$



Consider $g : [0,1] \to \mathbb{R}$ $\quad g(x) = \begin{cases} -1 & x \text{ rational} \\ 1 & x \text{ irrational} \end{cases}$

- Consider a partition $P = \{x_i\}_0^n$ on $[0,1]$
- In any interval $[x_{i-1}, x_i]$, $\exists$ irrational & rational
- real numbers

$\therefore$ $m_i = \inf\limits_{[x_{i-1}, x_i]} g(x) = -1$, $\quad M_i = \sup\limits_{[x_{i-1}, x_i]} g(x) = +1$

$\therefore$ $L(g,P) = \sum_{i=1}^{n} m_i (x_i - x_{i-1}) = \sum_{i=1}^{n} (-1)(x_i - x_{i-1}) = -(b-a) \neq 0$

$U(g,P) = \sum_{i=1}^{n} M_i (x_i - x_{i-1}) = \sum_{i=1}^{n} (1)(x_i - x_{i-1}) = (b-a) \neq 0$

$\therefore$ $U(g,P) - L(g,P) = 2(b-a) > 0$ $\quad b \neq a$ $\quad \forall P$.

$\therefore$ Given $\varepsilon > 0$, $\varepsilon = (b-a)$, $\nexists P$ s.t. $U(g,P) - L(g,P) < \varepsilon$

$\therefore$ RIC fails

$\therefore$ g not Riemann integrable.

$G(x) = \int_a^{x^2} h(t)\,dt$ does $G'(x)$ exist, and if so what is its value?

$G(y) = \int_a^y h(t)\,dt$

$y \in [a, b^2]$

Method 1

FTC: $H(y) = \int_a^y h(t)\,dt$

then $H'(y) = h(y)$.

Now $G(x) = H(x^2) = H(y(x))$ composed fn.

i.e. $G(x) = H(y)$ and $y(x) = x^2$

$H'(y) = h(y)$, $y'(x) = 2x$.

$\therefore\ G'(x) = H'(y)\,y' = h(x^2)\cdot 2x$

Method 2

$$G'(x) = \lim_{k\to 0} \frac{G(x+k) - G(x)}{k}$$

$$= \lim_{k\to 0} \frac{1}{k}\left[\int_a^{(x+k)^2} h(t)\,dt - \int_a^{x^2} h(t)\,dt\right]$$

$$= \lim_{k\to 0} \frac{1}{k} \int_{x^2}^{(x+k)^2} h(t)\,dt, \quad h \text{ continuous}$$

↓ MVThm

$$= \lim_{k\to 0} \frac{1}{k}\, h(c_k)\left[(x+k)^2 - x^2\right], \quad c_k \in [x^2, (x+k)^2]$$

$$= \lim_{k\to 0} h(c_k) \frac{\left[\cancel{x^2} + 2xk + k^2 - \cancel{x^2}\right]}{k}$$

$$= \lim_{k\to 0} h(c_k)(2x + k), \quad \text{where } c_k \to x^2, \text{ as } k \to 0$$

MVThm for integrals
$\int_A^B f(x)\,dx = f(c)(B-A)$

But h is continuous $\therefore\ h(c_k) \to h(x^2)$, as $k \to 0$.

$$\therefore\ \frac{G(x+k) - G(x)}{k} \to h(x^2)\cdot 2x \text{ as } k \to 0$$

i.e. $G'(x)$ exists and $G'(x) = h(x^2)\cdot 2x$