TUTORIAL 10

(1) Discussion of problem 1 of the Test

(2) An example of the use of Lagrangian mechanics: the 2-body problem (setup in the Newtonian theory)

(1) See the solutions available under "Week 8"

(2) Consider two particles of mass m_1, m_2.

The total kinetic energy is

$$T = \frac{1}{2} m_1 \left(\frac{d\underline{x}_1}{dt}\right)^2 + \frac{1}{2} m_2 \left(\frac{d\underline{x}_2}{dt}\right)^2$$

while the gravitational potential energy is

$$U = -G \frac{m_1 m_2}{|\underline{x}_1 - \underline{x}_2|}$$

It is convenient to introduce the variables

$$\underline{X}_{CM} = \frac{m_1 \underline{x}_1 + m_2 \underline{x}_2}{(m_1 + m_2)} \quad \text{(\underline{centre} of \underline{mass} \underline{position})}$$

and $\underline{x} \equiv \underline{x}_1 - \underline{x}_2$ (<u>relative distance</u>).

Then let us rewrite T (since U already depends only on $\underline{x}$). By defining

total mass $m = m_1 + m_2$; reduced mass $\mu = \frac{m_1 m_2}{m}$,

we have

$$\frac{1}{2} m \left(\frac{d \underline{X}_{cm}}{dt} \right)^2 + \frac{1}{2} \mu \left(\frac{d \underline{x}}{dt} \right)^2 =$$

$$\frac{1}{2} \left\{ (m_1 + m_2) \left(\frac{m_1}{m} \frac{d\underline{x}_1}{dt} + \frac{m_2}{m} \frac{d\underline{x}_2}{dt} \right)^2 + \right.$$

$$\left. \frac{m_1 m_2}{m} \left(\frac{d\underline{x}_1}{dt} - \frac{d\underline{x}_2}{dt} \right)^2 \right\} =$$

$$\frac{1}{2} \left\{ \left(\frac{m_1^2}{m} + \frac{m_1 m_2}{m} \right) \left(\frac{d\underline{x}_1}{dt} \right)^2 + \left(\frac{m_2^2}{m} + \frac{m_1 m_2}{m} \right) \left(\frac{d\underline{x}_2}{dt} \right)^2 \right\} = T$$

Notice that the cross terms $\sim \frac{d\underline{x}_1}{dt} \cdot \frac{d\underline{x}_2}{dt}$ vanish.

Thus we can write

$$T = \frac{1}{2} m \left(\frac{d \underline{X}_{cm}}{dt} \right)^2 + \frac{1}{2} \mu \left(\frac{d\underline{x}}{dt} \right)$$

Thene we have the following Lagrangian

$$L = T - V = \frac{1}{2} m \left(\frac{d \underline{X}_{cm}}{dt} \right)^2 + \frac{1}{2} \mu \left(\frac{d\underline{x}}{dt} \right)^2 + \frac{G M \mu}{|\underline{x}|}$$

The first term is decoupled from the others, so the centre of mass $\underline{X}_{CM}$ follows the equations of motion of a free particle!

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\left(\frac{dX^i_{CM}}{dt}\right)}\right) - \frac{\partial L}{\partial X^i_{CM}} = \frac{d}{dt}\left(\frac{\partial L}{\partial\left(\frac{dX^i_{CM}}{dt}\right)}\right) = 0 \Rightarrow$$

$$\frac{\partial L}{\partial\left(\frac{dX^i_{CM}}{dt}\right)} = p^i_{CM} \leftarrow \text{constant} \qquad m\frac{dX^i_{CM}}{dt} = P^i_{CM}$$

Clearly the value of P^i_{CM} depends on the reference frame. If the observer is co-moving with $\underline{X}_{CM}$ then $\underline{P}_{CM} = 0$

This is the centre of mass frame.

Focusing on $L_{rel} = \frac{1}{2}\mu\left(\frac{dx}{dt}\right)^2 + \frac{GM\mu}{|x|}$ we can study the relative motion of the two objects. L_{rel} shows that, in the Newtonian theory, it is equivalent to the motion of a single object (whose position is $\underline{x}$ and mass is μ) in an effective gravitational potential

$$\phi_{eff} = -\frac{GM}{|\underline{x}|}$$