Week 10 Thursday lecture

PLAN: (1) Gravitational waves (introduction)

(2) Linearised Einstein equations

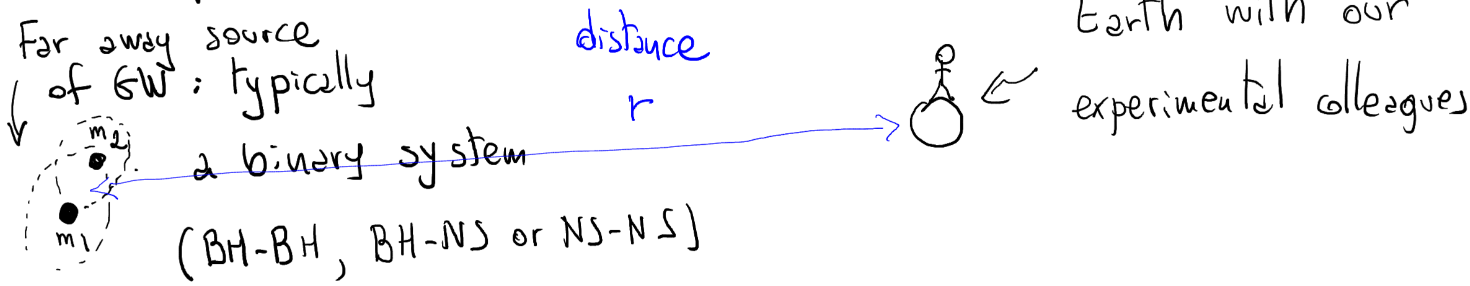
(1) One of the key prediction of GR is the existence of gravitational waves, i.e. time-dependent perturbations of space-time which propagate energy (very much as electromagnetic waves are perturbations of the electric and magnetic field). The subject started with two papers by Einstein in 1916-1918 and has a complicated history... see Box 11.2 pag. 553 of the book "Gravity" by Poisson and Will (Cambridge Univ. press) or D. Kennefick "Travelling at the speed of thought: Einstein and the quest for Gravitational Waves". On the experimental side let us just recall two major breakthroughs

- The discovery of the Hulse-Taylor pulsar which provided the first indirect but clear evidence of the existence of Gravitational Waves (1974)

- The first direct detection by LIGO (announced in 2016, but the event took place in 2015) of a GW emitted by a black hole binary system inspiraling and then merging. Many more such events have been detected in this new era of GW astronomy

The goal of this final part of the course is to we the knowledge developed so far to understand the two milestones mentioned above

The qualitative picture is the following



We expect a very weak perturbation of the metric because the source is far away.. Thus it makes sense to linearise $g_{\mu\nu}$ (as detected on the earth) and look at the time-dependent perturbation around the local static metric. Schematically

$$g_{ab} \sim \eta_{ab} + \frac{G\mu}{c^2 r}\, \varkappa\, h_{ab}$$

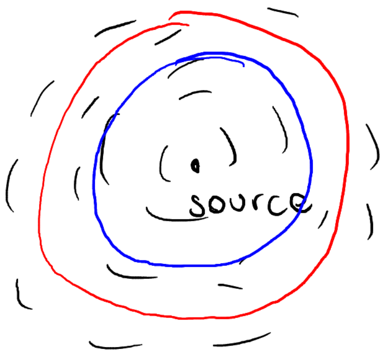
parameter depending on the relative distance of m_1, m_2

static part approximated to η_{ab}

reduced mass of the binary $\mu = \frac{m_1 m_2}{(m_1+m_2)}$

Comments:

- the $1/r$ fall off follow just from energy conservation and the fact that we are considering GW propagating in 3D space



Flux of Energy $\sim \left(\frac{dh}{dt}\right)^2$

It should be the same through the blue and red spheres

$$\left(\frac{\partial h}{\partial t}\right)^2_{r=R_{blue}} \cdot 4\pi R^2_{blue} = \left(\frac{\partial h}{\partial t}\right)^2_{r=R_{red}} 4\pi R^2_{red} \Rightarrow h \sim \frac{1}{r}$$

- we should define precisely what we mean by h_{ab}. In the formulae above it is <u>not</u> an observable quantity as it depends on the coordinate system

- We are interested both in the amplitude of the wave and in its phase $h \sim e^{i\omega(t)t}$

- The energy emitted comes from the potential energy of the binary, which means that the orbit shrinks over time till the two body merge (in a cataclysmic event!) and form a single new object which "rings" down to equilibrium

Let me conclude with the following picture from the LIGO paper announcing the first GW direct detection

Selected for a Viewpoint in *Physics*

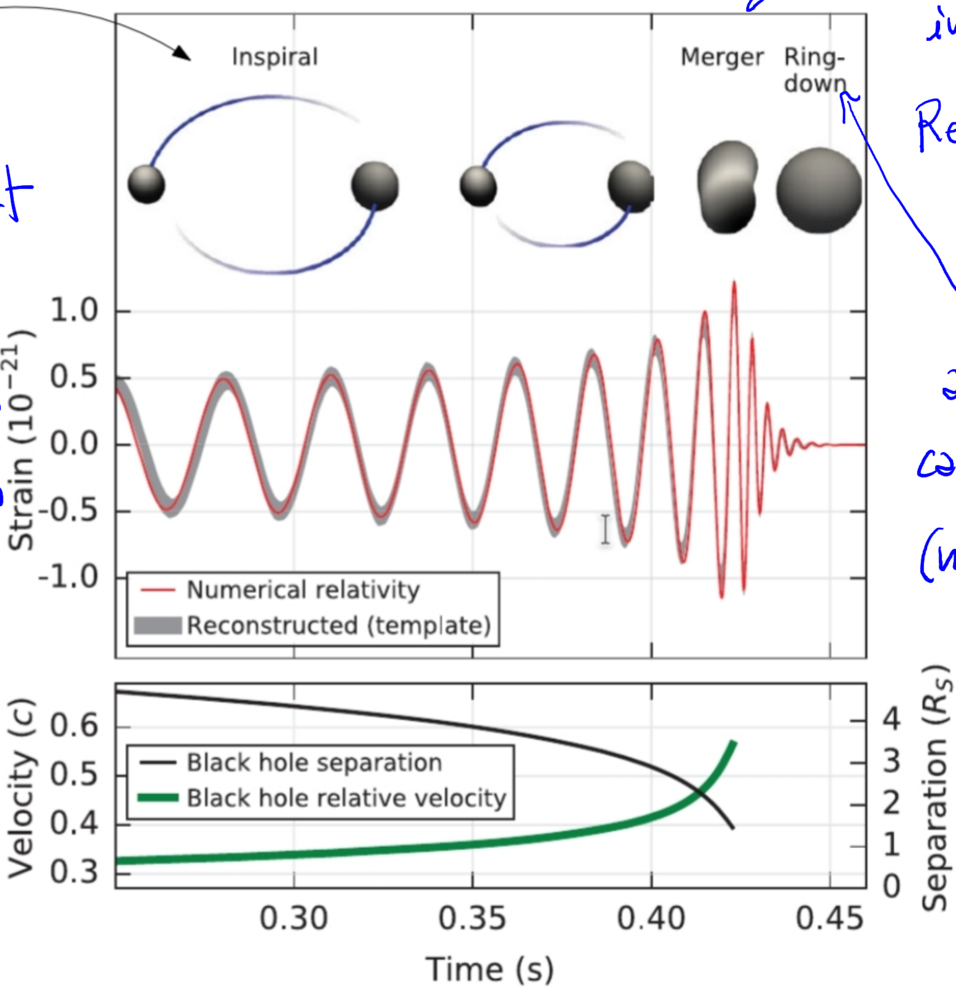
PRL **116,** 061102 (2016) PHYSICAL REVIEW LETTERS week ending 12 FEBRUARY 2016

Observation of Gravitational Waves from a Binary Black Hole Merger

B. P. Abbott *et al.*[*]
(LIGO Scientific Collaboration and Virgo Collaboration)
(Received 21 January 2016; published 11 February 2016)

Hard problem: to be addressed in Numerical Relativity

We focus on this part within the Post-Newtonian approximation



Another analytic approx. can be used here (not covered in this course)

(2) With this motivation in mind let us study the linearised Einstein equation. We start from

$$g_{ab}(x) = \eta_{ab} + h_{ab}(x)$$

we consider h_{ab} to be small, so any term $\sim h^2$ is neglected

Then $g^{ab}(x) \simeq \eta^{ab} - \underbrace{\left(\eta^{ac}\eta^{bd} h_{cd}\right)}_{\equiv h^{ab}}$

Check $g^{ae} g_{eb} = \left[\eta^{ae} - \eta^{ac}\eta^{ed} h_{cd}\right]\left[\eta_{eb} + h_{eb}\right] \simeq$

$$\eta^{ae}\eta_{eb} + \eta^{ae} h_{eb} - \eta^{ac}\underbrace{\eta^{de}\eta_{eb}} h_{cd} + \dots \simeq$$

$$\delta^a{}_b + \eta^{ae} h_{eb} - \eta^{ac}\delta^d{}_b h_{cd} + \dots \simeq \delta^a{}_b$$

where in the second step we neglect term that are quadratic in h. Thus we see that, within this approximation, we can raise/lower indices by using the Minkowski metric (see the Eq. at the top of this page).

Let's now linearise the Riemann tensor

$$R^d{}_{acb} = \frac{\partial \Gamma^d_{ba}}{\partial x^c} - \frac{\partial \Gamma^d_{ca}}{\partial x^b} + \Gamma^d_{ce}\Gamma^e_{ba} - \Gamma^d_{be}\Gamma^e_{ca}$$

and since $\Gamma^a_{bc} = \frac{1}{2} g^{ad}\left(\frac{\partial g_{bd}}{\partial x^c} + \frac{\partial g_{cd}}{\partial x^b} - \frac{\partial g_{bc}}{\partial x^d}\right)$

$$\simeq \frac{1}{2}\eta^{ad}\left(\frac{\partial h_{bd}}{\partial x^c} + \frac{\partial h_{cd}}{\partial x^b} - \frac{\partial h_{bc}}{\partial x^d}\right)$$

So $\Gamma \sim h$ and we can neglect the Γ^2 in the

Riemann tensor above

$$R^d{}_{acb} \simeq \frac{\partial \Gamma^d_{ba}}{\partial x^c} - \frac{\partial \Gamma^d_{ca}}{\partial x^b}$$

Then the linearisd Ricci tensor reads

$$R^c{}_{acb} = R_{ab} \simeq \frac{\partial \Gamma^c_{ba}}{\partial x^c} - \frac{\partial \Gamma^c_{ca}}{\partial x^b}$$

We define $h = h^a{}_a = \eta^{ab} h_{ab}$ and have

$$R_{ab} \simeq \frac{1}{2}\partial_b \partial^c h_{ac} + \frac{1}{2}\partial_a \partial^c h_{bc} - \frac{1}{2}\partial^c \partial_c h_{ab} - \frac{1}{2}\partial_a \partial_b h$$

and for the Ricci scalar

$$R \simeq \partial^d \partial^c h_{dc} - \partial^c \partial_c h \,,$$ thus the Einstein tensor reads

$$G_{ab} \simeq \frac{1}{2}\partial_b \partial^c h_{ac} + \frac{1}{2}\partial_a \partial^c h_{bc} - \frac{1}{2}\partial^c \partial_c h_{ab} - \frac{1}{2}\partial_a \partial_b h - \frac{1}{2}\eta_{ab}\left(\partial^d \partial^c h_{dc} - \partial^c \partial_c h\right)$$

This can be written in a more compact form by introducing $\bar{h}_{ab} \equiv h_{ab} - \frac{1}{2}\eta_{ab} h$, see the typewritten notes. In order to simplify this complicated expression we exploit this possibility to change the form of the metric by performing a coordinate transformation. In the spirit of this analysis we linearise this step as well $x^a \to x^a + \xi^a(x)$ where ξ^a is small. Then

$$g_{ab}\, dx^a dx^b = (\eta_{ab} + h_{ab}(x))\, dx^a dx^b \longrightarrow$$

$$(\eta_{ab} + h_{ab}(x + \cancel{\xi}))\, d(x^a + \xi^a)\, d(x^b + \xi^b)$$

where we can ignore this since it would yield terms $\sim \xi\, \partial h$ and so quadratic in the small quantities. Thus

$$(\eta_{ab} + h_{ab}(x))\, dx^a dx^b \to (\eta_{ab} + h_{ab} + \partial_a \xi_b + \partial_b \xi_a)\, dx^a dx^b$$

where we used $d\xi_c = \dfrac{\partial \xi_c}{\partial x^a} dx^a = \partial_a \xi_c\, dx^a$. Thus

we see that $h_{ab} \to h_{ab} + \partial_a \xi_b + \partial_b \xi_a$ under change of coordinates and we can use this freedom to have

$$\partial^a \left(h_{ab} - \frac{1}{2}\eta_{ab} h\right) = 0 \qquad \text{harmonic or De Donder gauge}$$

[Comment: suppose that $\partial^a (h_{ab} - \frac{1}{2}\eta_{ab} h) = f_b \neq 0$, then under coordinate transformation we get to a new h_{ab} satisfying

$$f_b + \partial^a \partial_a \xi_b + \cancel{\partial_b (\partial^a \xi_a)} - \cancel{\frac{1}{2}\partial_b (2\partial^a \xi_a)} = 0 \leftarrow \text{Wave equation for } \xi$$

Thus we can choose ξ_b to cancel f_b and go to the harmonic gauge $\rfloor$

In this gauge G_{ab} simplifies since $\partial^b h_{ab} = \frac{1}{2}\partial_a h$

$$G_{ab} \simeq \underbrace{\frac{1}{2}\partial_b \partial^c h_{ac}}_{\cancel{\frac{1}{4}\partial_b\partial_a h}} + \underbrace{\frac{1}{2}\partial_a \partial^c h_{bc}}_{\cancel{\frac{1}{4}\partial_a\partial_b h}} - \frac{1}{2}\partial^c\partial_c h_{ab} - \cancel{\frac{1}{2}\partial_a\partial_b h} -$$

$$\frac{1}{2}\eta_{ab}\Big(\underbrace{\partial^d\partial^c h_{dc}}_{\frac{1}{2}\partial^c\partial_c h} - \partial^c\partial_c h\Big) =$$

$$-\frac{1}{2}\partial^c\partial_c h_{ab} + \frac{1}{4}\eta_{ab}\partial^c\partial_c h = -\frac{1}{2}\partial^c\partial_c \bar{h}_{ab}$$

Thus the linearised Einstein equations read

$$G_{ab} = 8\pi G T_{ab} \Rightarrow \boxed{\partial^c\partial_c \bar{h}_{ab} \simeq -16\pi G T_{ab}}$$