2

a) By direct computation,

$$\nabla_a \nabla^a f = \partial_a \nabla^a f + \Gamma^a{}_{ab} \nabla^b f$$

From the definition of Christoffel symbols, we have

$$\Gamma^a{}_{ab} = \frac{1}{2} g^{ac} \left(\cancel{\partial_a g_{bc}} + \partial_b g_{ac} - \cancel{\partial_c g_{ab}} \right) =$$

$$= \frac{1}{2} g^{ac} \partial_b g_{ac}$$

To compute this, recall that for any matrix M

$$\mathrm{Tr}\left(M^{-1}(x) \frac{\partial}{\partial x^a} M(x) \right) = \frac{\partial}{\partial x^a} \ln \det M(x)$$

To show this, consider the variation in $\ln \det M$

due to a variation δx^a in x^a:

$$\delta \ln \det M = \ln \det (M + \delta M) - \ln \det M$$

$$= \ln \frac{\det (M + \delta M)}{\det M}$$

$$= \ln \det M^{-1} (M + \delta M)$$

$$= \ln \det (\mathbb{1} + M^{-1} \delta M)$$

$$= \ln (\mathbb{1} + \mathrm{Tr}\, M^{-1} \delta M) + O(\delta M^2)$$

$$= \mathrm{Tr}\, M^{-1} \delta M + O(\delta M^2)$$

Taking the coefficient of δx^a on both sides gives the desired result. Applying this result to the case when that matrix M is the metric g_{ab}, we find

$$\Gamma^a{}_{ab} = \frac{1}{2} g^{ac} \partial_b g_{ac}$$
$$= \frac{1}{2} \partial_b \ln g \quad = \frac{1}{\sqrt{|g|}} \partial_b \sqrt{|g|}$$

Plugging this result into the original equation,

$$\nabla_a \nabla^a f = \partial_a \nabla^a f + \Gamma^a{}_{ab} \nabla^b f =$$
$$= \partial_a \left(g^{ab} \partial_b f \right) + \frac{1}{\sqrt{|g|}} \partial_b (\sqrt{|g|})\, g^{bc} \partial_c f$$
$$= \frac{1}{\sqrt{|g|}} \partial_a \left(\sqrt{|g|}\, g^{ab} \partial_b f \right)$$

which is the desired result.

b) We write the flat metric on $\mathbb{R}^n$ in spherical coordinates as

$$ds^2 = dr^2 + r^2 d\Omega^2_{(n-1)}$$

where $d\Omega^2_{(n-1)}$ is the metric on the unit

$(n-1)$-sphere; its actual form is not relevant here, the important point is that it only depends on angles.

We compute the determinant of the metric as

$$\det g = r^{2(n-1)} \det(\Omega_{(n-1)})$$

Considering a spherically symmetric function on $\mathbb{R}^n$, $f = f(r)$, the Laplacian is:

$$\Delta f = \frac{1}{\sqrt{|g|}} \partial_a \left(\sqrt{|g|}\, g^{ab} \partial_b f \right) =$$

$$= \frac{1}{r^{n-1} (\det \Omega_{n-1})^{1/2}} \partial_a \left(r^{n-1} (\det \Omega_{n-1})^{1/2} g^{ar} \partial_r f \right)$$

$$= \frac{1}{r^{n-1}} \partial_r \left(r^{n-1} \partial_r f \right)$$

$$= \partial_r^2 f + \frac{n-1}{r} \partial_r f$$

In going from the 2nd line to the 3rd we used that $g^{ar} = \delta^{ar}$ so only the radial derivatives will contribute; since $\det \Omega_{n-1}$ only depends on the angles on the $(n-1)$-sphere, it won't be affected by the radial derivatives.

Now we can find all spherically symmetric solutions to the Laplace equation on $\mathbb{R}^n$:

$$\Delta f(r) = \partial_r^2 f + \frac{n-1}{r}\partial_r f = \frac{1}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r f\right) = 0$$

Integrating this equation once gives,

$$r^{n-1}\partial_r f = \tilde{C}_1 \Rightarrow \partial_r f = \frac{\tilde{C}_1}{r^{n-1}}$$

where $\tilde{C}_1$ = constant. Integrating again,

$$f(r) = \frac{C_1}{r^{n-2}} + C_2$$

where C_1 and C_2 are constants and we have redefined $C_1 = -\frac{\tilde{C}_1}{n-2}$.

Notice that this solution isn't valid for $n=2$. In this case, we have

$$\partial_r f = \frac{\tilde{C}_1}{r} \Rightarrow f(r) = \tilde{C}_1 \ln r + \tilde{C}_2$$
$$= \tilde{C}_1 \ln\left(r/C_2\right)$$

⑥ a) $$0 = G_{ab} + \Lambda g_{ab}$$
$$= R_{ab} - \frac{1}{2} R g_{ab} - \frac{3}{L^2} g_{ab}$$

Taking the trace of this equation yields

$$0 = R - 2R - \frac{12}{L^2} \Rightarrow R = -\frac{12}{L^2}$$

Substituting this into the Einstein eq:

$$0 = R_{ab} - \frac{1}{2} R g_{ab} - \frac{3}{L^2} g_{ab} =$$

$$= R_{ab} - \frac{1}{2}\left(-\frac{12}{L^2}\right) g_{ab} - \frac{3}{L^2} g_{ab}$$

$$= R_{ab} + \frac{3}{L^2} g_{ab}$$

b) Let us define a new tensor:

$$E_{ab} = R_{ab} + \frac{3}{L^2} g_{ab} = 0$$

We solve the Einstein equations following the same steps as in the lectures to find the Schwarzschild metric in the vacuum case:

Consider the combination of the Einstein equation:

$$e^{2(B-A)} E_{tt} + E_{rr} = \frac{2}{r}(A' + B') = 0$$

This implies $A(r) = -B(r)$. Recall that the integration constant can be set to zero by rescaling the time coordinate t.

Considering the $E_{\theta\theta} = 0$ equation and using the previous result yields

$$\partial_r (r e^{2A}) = 1 + \frac{3r^2}{L^2}$$

$$\Rightarrow \quad r e^{2A} = r + \frac{r^3}{L^2} + C \quad , \quad C = \text{constant}$$

$$\Rightarrow \quad e^{2A} = 1 + \frac{r^2}{L^2} + \frac{C}{r}$$

Demanding that in the $L \to \infty$ limit we recover the usual asymptotically flat Schwarzschild solution fixes $C = -2GM$. Hence, the metric is:

$$ds^2 = -\left(1 + \frac{r^2}{L^2} - \frac{2GM}{r}\right) dt^2 + \frac{dr^2}{1 + \frac{r^2}{L^2} - \frac{2GM}{r}} + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$$

(5) Question: Solution

$$ds^2 = -\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)dt^2 + \frac{dr^2}{1-\frac{2M}{r}+\frac{Q^2}{r^2}} + r^2(d\theta^2+\sin^2\theta\, d\phi^2)$$

$$\mathcal{L} = -\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)\dot{t}^2 + \frac{\dot{r}^2}{1-\frac{2M}{r}+\frac{Q^2}{r^2}} + r^2(\dot{\theta}^2+\sin^2\theta\,\dot{\phi}^2)$$

$$E = -K_a \dot{x}^a = \left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)\dot{t}$$

$$L = R_c \dot{x}^c = r^2\dot{\phi}$$

$$\mathcal{L} = -\frac{E^2}{(\;\;)} + \frac{\dot{r}^2}{(\;\;)} + \frac{L^2}{r^2} = -\varepsilon$$

$$-E^2 + \dot{r}^2 + \left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)\left(\frac{L^2}{r^2}+\varepsilon\right) = 0$$

$$\frac{1}{2}\dot{r}^2 + \underbrace{\frac{1}{2}\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)\left(\frac{L^2}{r^2}+\varepsilon\right)}_{V(r)} = \mathcal{E}, \quad \mathcal{E} = \frac{1}{2}E^2$$

$M^2 > Q^2$

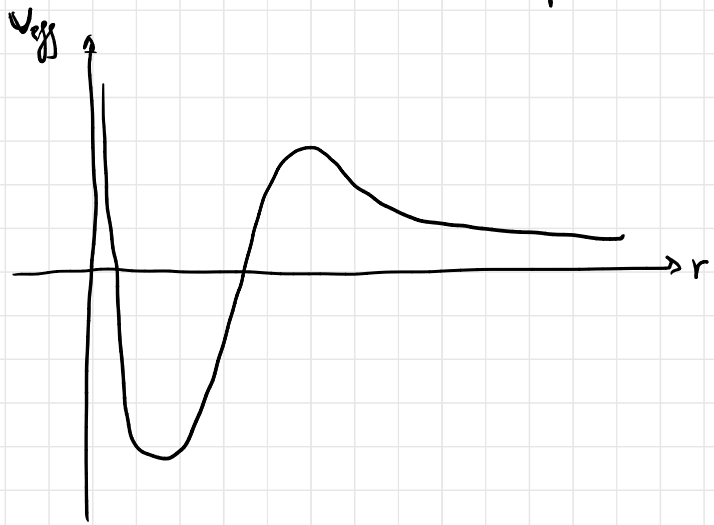
$$V(r) = \frac{1}{2}\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)\left(\frac{L^2}{r^2}+\varepsilon\right)$$

For null geodesics, $V_{eff} = \frac{L^2}{2r^2}\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)$

$V_{eff} = 0$ for $r_{\pm} = M \pm \sqrt{M^2-Q^2}$

$V'_{eff} = 0$ for $r = \frac{1}{2}\left(3M \pm \sqrt{9M^2 - 8Q^2}\right)$

So V_{eff} looks as follows



→ Students should identify:

i) $V(r) \searrow 0$ for $r \to \infty$

ii) $V(r) \to +\infty$ for $r \to 0$

iii) There is a maximum and a minimum.

(5) The metric is:

$$ds^2 = -\left(\frac{r^2}{\ell^2} - \frac{r_0^2}{r^2}\right) dt^2 + \left(\frac{r^2}{\ell^2} - \frac{r_0^2}{r^2}\right)^{-1} dr^2 + r^2 \left(dx^2 + dy^2 + dz^2\right)$$

from which it follows that the Lagrangian governing the geodesics is

$$\mathcal{L} = -\left(\frac{r^2}{\ell^2} - \frac{r_0^2}{r^2}\right) \dot{t}^2 + \left(\frac{r^2}{\ell^2} - \frac{r_0^2}{r^2}\right)^{-1} \dot{r}^2 + r^2\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

a) Using the definitions in the notes,

$$E = -g_{ab} (\partial_t)^a \dot{x}^b = \left(\frac{r^2}{\ell^2} - \frac{r_0^2}{r^2}\right) \dot{t}$$

$$K_x = g_{ab} (\partial_x)^a \dot{x}^b = r^2 \dot{x}$$

$$K_y = g_{ab} (\partial_y)^a \dot{x}^b = r^2 \dot{y}$$

$$K_z = g_{ab} (\partial_z)^a \dot{x}^b = r^2 \dot{z}$$

These quantities can also be calculated using that $\mathcal{L}$ does not depend on t, x, y and z, so

$$\frac{\partial \mathcal{L}}{\partial \dot{t}} = -2\left(\frac{r^2}{\ell^2} - \frac{r_0^2}{r^2}\right) \dot{t} = -2E = \text{const}$$

and so on.

b) Substituting the results from a) into $\mathcal{L}$ we get

$$\mathcal{L} = -\frac{E^2}{\frac{r^2}{\ell^2} - \frac{r_0^2}{r^2}} + \frac{\dot{r}^2}{\frac{r^2}{\ell^2} - \frac{r_0^2}{r^2}} + \frac{1}{r^2}\left(K_x^2 + K_y^2 + K_z^2\right) = -\varepsilon$$

$$\Rightarrow \frac{1}{2}\dot{r}^2 + \frac{1}{2}\left(\frac{r^2}{\ell^2} - \frac{r_0^2}{r^2}\right)\left(\varepsilon + \frac{\vec{K}^2}{r^2}\right) = \frac{1}{2}E^2$$

$$\Rightarrow V(r) = \frac{1}{2}\left(\frac{r^2}{\ell^2} - \frac{r_0^2}{r^2}\right)\left(\varepsilon + \frac{\vec{K}^2}{r^2}\right)$$

when $\vec{K}^2 = K_x^2 + K_y^2 + K_z^2$ and $\varepsilon = 0, 1$ for null and timelike geodesics.

Massive particles $\rightarrow$ timelike geodesics $(\varepsilon = 1)$

$$V(r) = \frac{1}{2}\left(\frac{r^2}{\ell^2} - \frac{r_0^2}{r^2}\right)\left(1 + \frac{\vec{K}^2}{r^2}\right)$$

$$V'(r) = \frac{r}{\ell^2} + \frac{r_0^2}{r^3} + \frac{2\vec{K}^2 r_0^2}{r^5} > 0$$

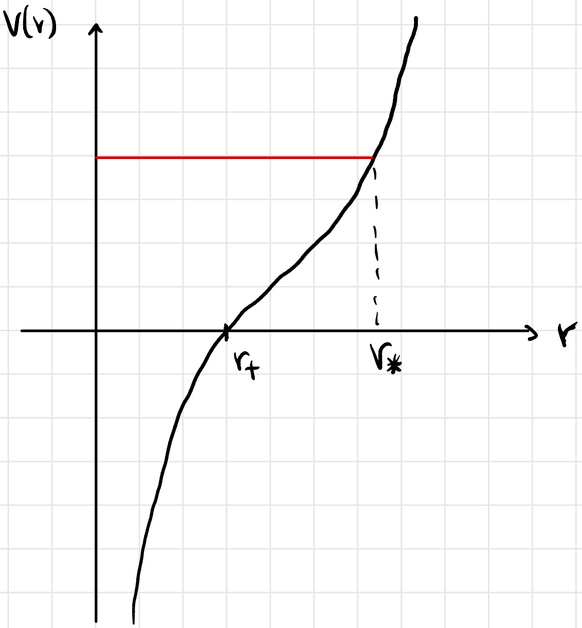
$\Rightarrow$ $V(r)$ doesn't have extrema

$$V(r) \sim -\frac{r_0^2 \vec{K}^2}{2r^4} \quad \text{as} \quad r \to 0$$

$$V(r) \sim \frac{r^2}{2\ell^2} \quad \text{as} \quad r \to \infty$$

$V(r) = 0$ @ $\frac{r^2}{\ell^2} - \frac{r_0^2}{r^2} = 0 \Rightarrow r_t = (\ell r_0)^{1/2}$

Thus, $V(r)$ looks like



From the shape of $V(r)$ it follows that massive particles which initially travel towards $r \to \infty$ will reach a maximum $r = r_*$ given by

$$V(r_*) = \frac{1}{2} E^2$$

and then they'll bounce back and reach $r = 0$ in finite time. Particles which initially travel towards smaller r will inevitably reach $r=0$ in finite time.

c) Radial timelike geodesics ($\vec{k}=0$) are given by

$$\dot{r}^2 + \left(\frac{r^2}{\ell^2} - \frac{r_0^2}{r^2}\right) = E^2$$

where $\dot{r} = \frac{dr}{d\tau}$ and τ is the proper time.

$$\Rightarrow \quad \frac{dr}{d\tau} = -\sqrt{E^2 - \left(\frac{r^2}{\ell^2} - \frac{r_0^2}{r^2}\right)}$$

and we pick the "$-$" sign because the particle is travelling inwards. Then, the proper time taken is given by:

$$\Delta\tau = -\int_{r_*}^{0} dr \frac{1}{\sqrt{E^2 - \left(\frac{r^2}{\ell^2} - \frac{r_0^2}{r^2}\right)}}$$

$$= \frac{\ell}{2}\left[\arctan\left(\frac{E^2 - 2r_*^2/\ell^2}{2\sqrt{E^2 r_*^2/\ell^2 - r_*^4/\ell^4 + r_0^2/\ell^2}}\right) - \arctan\left(\frac{E^2}{2r_0/\ell}\right)\right]$$