MTH5113 (2023/24): Problem Sheet 9

All coursework should be submitted individually.

- Problems marked "[**Marked**]" should be submitted and will be marked.

Please submit the completed problem on QMPlus:

- At the portal for **Coursework Submission 2**.

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(1) *(Warm-up)*

(a) Consider the (real-valued) function

$$F : \mathbb{R}^3 \to \mathbb{R}, \qquad F(x, y, z) = xy^2z^3,$$

as well as the parametric surface

$$\mathbf{P} : (0, 1) \times (0, 1) \to \mathbb{R}^3, \qquad \mathbf{P}(u, v) = (1,\ u,\ v).$$

Compute the surface integral of F over $\mathbf{P}$.

(b) Consider the (real-valued) function

$$G : \mathbb{R}^3 \to \mathbb{R}, \qquad G(x, y, z) = x^2 + y^2,$$

as well as the parametric surface

$$\tau : (0, 2\pi) \times (0, 1) \to \mathbb{R}^3, \qquad \tau(u, v) = (v \cos u,\ v \sin u,\ v).$$

Compute the surface integral of G over τ.

(2) *(Intro to surface integrals)* One can also define an intermediate notion of surface integration of vector fields over *parametric surfaces*. More specifically:

Definition. Let $\sigma : U \to \mathbb{R}^3$ be a parametric surface, and let $\mathbf{F}$ be a vector field that is defined on the image of γ. We then define the *surface integral* of $\mathbf{F}$ over σ by

$$\int\int_\sigma \mathbf{F} \cdot d\mathbf{A} = \int\int_U \{\mathbf{F}(\sigma(u, v)) \cdot [\partial_1\sigma(u, v) \times \partial_2\sigma(u, v)]_{\sigma(u,v)}\} \, du dv.$$

(a) Consider the vector field $\mathbf{F}$ on $\mathbb{R}^3$ given by

$$\mathbf{F}(x,y,z) = \left(y,\ z^{5800} e^{x^{2000}+46y^{1523}},\ x\right)_{(x,y,z)},$$

and let $\mathbf{P}$ be the parametric plane

$$\mathbf{P} : (0,1) \times (0,1) \to \mathbb{R}^3, \qquad \mathbf{P}(u,v) = (1,\ u,\ v).$$

Compute the surface integral of $\mathbf{F}$ over $\mathbf{P}$.

(b) Consider the vector field $\mathbf{G}$ on $\mathbb{R}^3$ given by

$$\mathbf{G}(x,y,z) = (z,\ z,\ x^2 + y^2)_{(x,y,z)},$$

and let τ be the parametric torus

$$\tau : (0,2\pi) \times (0,1) \to \mathbb{R}^3, \qquad \tau(u,v) = (v\cos u,\ v\sin u,\ v).$$

Compute the surface integral of $\mathbf{G}$ over τ.

(c) Consider the vector field $\mathbf{H}$ on $\mathbb{R}^3$ given by

$$\mathbf{H}(x,y,z) = (-x,\ -y,\ z)_{(x,y,z)},$$

and let $\mathbf{q}$ be the (regular) parametric surface

$$\mathbf{q} : (0,1) \times (0,1) \to \mathbb{R}^3, \qquad \mathbf{q}(u,v) = (u,\ v,\ u^2 + v^2).$$

Compute the surface integral of $\mathbf{H}$ over $\mathbf{q}$.

(3) *(A Survey of Integration)* Let S denote the set

$$S = \{(u,v,u^2 - v^2) \in \mathbb{R}^3 \mid (u,v) \in (0,1) \times (0,1)\}.$$

(a) Show that S is a surface. In addition, give an injective parametrisation of S whose image is precisely all of S.

(b) Compute the surface integral over S of the real-valued function

$$F : \mathbb{R}^3 \to \mathbb{R}, \qquad F(x,y,z) = xy.$$

(The double integral you get from expanding the surface integral is not so pleasant; you will probably have to use the method of substitution twice to compute it.)

(c) Let us also assign to S the *upward-facing orientation*, i.e. the orientation in the *positive z-direction*. Then, compute the surface integral over S of the vector field

$$\mathbf{G}(x, y, z) = (xy^2,\ yx^2,\ 1)_{(x,y,z)}, \qquad (x, y, z) \in \mathbb{R}^3.$$

(4) [Marked] Let C denote the following disconnected surface:

$$C = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1 \cup x^2 + y^2 = 4,\ -1 < z < 1\}.$$

which describes a small cylinder surrounded by a larger cylinder. Moreover, let us orient $\mathcal{C}$ such that the outer cylinder has *outward* orientation and the inner cylinder is oriented *inwards*.

(a) Sketch this surface.

(b) Compute the surface integral over C of the function

$$G : \mathbb{R}^3 \to \mathbb{R}, \qquad G(x, y, z) = -\frac{x^2y^2}{66} + \frac{z^2}{8}.$$

Some useful hints: $2\sin(x)\cos(x) = \sin(2x)$ *and* $\sin^2(x) = \frac{1-\cos(2x)}{2}$.

(c) Compute the surface integral over C of the vector field $\mathbf{H}$ on $\mathbb{R}^3$ given by

$$\mathbf{H}(x, y, z) = (y, x, z)_{(x,y,z)},$$

(5) [Tutorial]

(a) Consider the surface *(you may assume this is indeed a surface)*

$$\mathcal{P} = \{(u, v, u^4 + v) \in \mathbb{R}^3 \mid (u, v) \in (0, 1) \times (-1, 1)\}.$$

Compute the surface integral over $\mathcal{P}$ of the following function:

$$F : \mathbb{R}^3 \to \mathbb{R}, \qquad F(x, y, z) = 6x^5.$$

(b) Consider the sphere,

$$\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\},$$

and let $\mathbb{S}^2$ be given the "outward-facing" orientation. Compute the surface integral over $\mathbb{S}^2$ of the vector field $\mathbf{F}$ on $\mathbb{R}^3$ defined by the formula

$$\mathbf{F}(x, y, z) = (0, 0, z^3)_{(x,y,z)}.$$

(6) *(A-levels, revisited)*

(a) Show that the surface area of a *sphere of radius* $r > 0$,

$$S_r = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = r^2\},$$

is equal to $4\pi r^2$.

(b) Show that the area of the side of a cone with base radius $r > 0$ and height $h > 0$,

$$C_{r,h} = \left\{ (x, y, z) \in \mathbb{R}^3 \mid 0 < z < h,\ x^2 + y^2 = r^2 \left(1 - \frac{z}{h}\right)^2 \right\},$$

is equal to $\pi r \sqrt{r^2 + h^2}$.

(7) *(Reversal of orientations)* Let $S \subseteq \mathbb{R}^3$ be an oriented surface, and let $\sigma : U \to S$ be a parametrisation of S. Moreover, define the set

$$U_r = \{(v, u) \mid (u, v) \in U\}$$

and define the parametric surface

$$\sigma_r : U_r \to \mathbb{R}^3, \qquad \sigma_r(v, u) = \sigma(u, v).$$

In other words, σ_r is precisely σ but with the roles of u and v reversed.

(a) Show that for any $(u, v) \in U$,

$$\partial_1 \sigma_r(v, u) \times \partial_2 \sigma_r(v, u) = -[\partial_1 \sigma(u, v) \times \partial_2 \sigma(u, v)].$$

(b) Show that σ_r is also a parametrisation of S, and that σ_r has the same image as σ.

(c) Use the formula from part (a) to conclude that if σ generates an orientation O of S, then σ_r generates the orientation opposite to O.

(8) *(The paradox of Gabriel's horn)* Consider the surface of revolution

$$G = \left\{ (x, y, z) \in \mathbb{R}^3 \middle| y^2 + z^2 = \frac{1}{x^2}, \ x > 1 \right\},$$

which is sometimes nicknamed *Gabriel's horn*. *(Before proceeding, you should search for "Gabriel's horn" on Google Images to see an illustration of* G*.)*

(a) Show that G has infinite surface area.

(b) Show that the interior of G,

$$I = \left\{ (x, y, z) \in \mathbb{R}^3 \middle| y^2 + z^2 \leq \frac{1}{x^2}, \ x > 1 \right\},$$

has finite volume.

In other words, *you can fill up the inside of the "horn" with a finite amount of paint, but you cannot paint the "horn" itself using a finite amount of paint*!