RELATIVITY – MTH6132

PROBLEM SET 10

1. Timelike geodesics in Schwarzschild.

(a) Derive equation for the radial motion of timelike geodesics as a function of the angle ϕ, equation (6.50) in the notes:

$$\left(\frac{dr}{d\phi}\right)^2 + \frac{r^4}{L^2} - \frac{2GM}{L^2}\, r^3 + r^2 - 2GM\, r = \frac{2\mathcal{E}}{L^2}\, r^4 \,.$$

(b) Defining a new variable $u = \frac{L^2}{GMr}$, show that u satisfies

$$\frac{d^2u}{d\phi^2} - 1 + u = \frac{3G^2M^2}{L^2}\, u^2 \,.$$

2. Null geodesics in Schwarzschild. Derive equation (6.67) in the notes for the radial motion of the null geodesics in the Schwarzschild geometry in terms of the variable $u = \frac{1}{r}$:

$$\frac{d^2u}{d\phi^2} + u = 3GMu^2 \,.$$

3. Consider a satellite orbiting a central mass M in a circle of Schwarzschild radius D (i.e., in a circular orbit of radius $r = D$ lying in the plane defined by $\theta = \pi/2 =$ constant). Show that the time to complete one revolution as measured by the t coordinate is given by

$$t = 2\pi \left(\frac{D^3}{GM}\right)^{1/2}$$

How would you calculate the period in terms of the proper time τ?

4. Shapiro time delay. Consider null geodesics on the equatorial plane ($\theta = \frac{\pi}{2}$) in the Schwarzschild spactime

$$-\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2\, d\phi^2 = 0$$

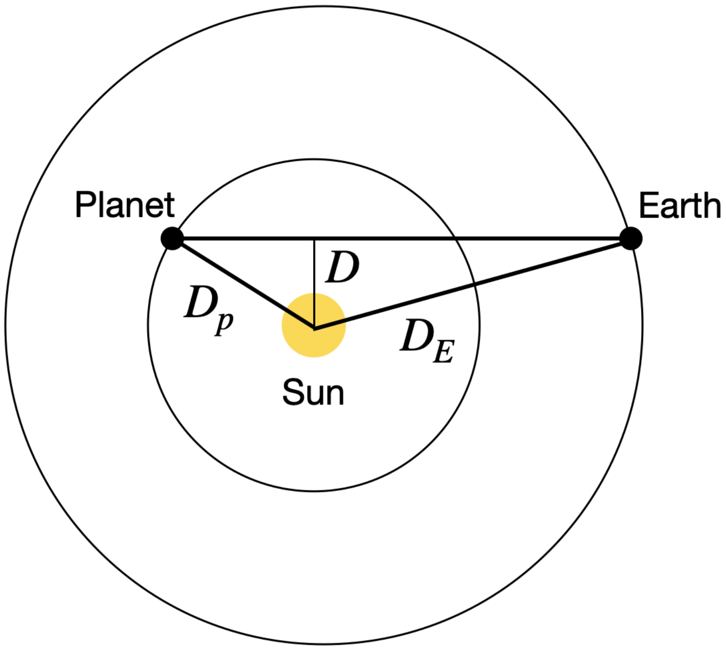
Since we are going to work to first order in GM/r, it is sufficient to assume that the photon's trajectory is a straight-line

$$r \sin\phi = D$$

where D is a constant.

(a) Find $d\phi$ in terms of dr for the above straight line trajectory.

(b) Use the previous results to solve for dt for a null ray to first order in GM/r.



(c) We are interested in the computing the travel time for a signal between a planet and the Earth. Compute such a travel time integrating the equation for dt. See figure below. (*Hint:* use a new variable u where $r = D\cosh u$.)

The experimental verification of the time delay consists in sending pulsed radar signals from Earth to Venus and mercury and timing the echoes as the positions of Earth and the planet change relative to the Sun. For Venus, the measured time delay is about 200 μs.

5. Equatorial null orbits in the Kerr black hole spacetime. The Kerr metric represents a rotating black hole in vacuum:

$$ds^2 = -\left(1 - \frac{2GMr}{\Sigma}\right) dt^2 - \frac{4GM\,a\,r\sin^2\theta}{\Sigma}\,dt\,d\phi + \frac{\Sigma}{\Delta}\,dr^2$$
$$+\,\Sigma\,d\theta^2 + \frac{\sin^2\theta}{\Sigma}\left[(r^2+a^2)^2 - a^2\Delta\,\sin^2\theta\right] d\phi^2$$

where $\Delta = r^2 - 2GMr + a^2$ and $\Sigma = r^2 + a^2\cos^2\theta$. Here M and a constants corresponding to the mass and the angular momentum per unit mass (so the angular momentum is $J = Ma$). You may assume $M > a \geq 0$.

(a) Show that the line element above reduces to the Schwarzschild spacetime for $a = 0$.

(b) Find the Largrangian controlling the equatorial (i.e., $\theta = \frac{\pi}{2}$) orbits for null rays.

(c) Find the equation governing the radial motion of the null geodesics.

(d) Assuming that $(r^2+a^2)^2 - a^2\Delta\,\sin^2\theta > 0$, show that the orbit of the photon in the equatorial plane cannot have a turning point inside $r = r_+ = GM + \sqrt{(GM)^2 - a^2}$. This means that ingoing light rays cannot escape once they cross r_+, so this is the event horizon.

6. Equatorial timelike orbits in the Kerr black hole spacetime.

(a) Find the Largrangian controlling the equatorial (i.e., $\theta = \frac{\pi}{2}$) orbits for the timelike geodesics in the Kerr spacetime.

(b) Find the effective potential controlling the radial motion of the geodesics.

(c) Setting $M = 1$ for simplicity, sketch the effective potentials for the timelike and null cases and describe the possible orbits.

7. Write the Kerr metric in ingoing EF coordinates. On the equatorial plane, at which value of r do the outgoing radial light cones tilt towards smaller values of r?

8. The Schwarzschild anti-de Sitter spacetime is given by the following line element:

$$ds^2 = -\left(1 - \frac{2GM}{r} + \frac{r^2}{\ell^2}\right) dt^2 + \left(1 - \frac{2GM}{r} + \frac{r^2}{\ell^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

(a) Find the Largrangian governing both the timelike and null geodesics.

(b) Find the effective potential controlling the radial motion of both timelike and null geodesics.

(c) Sketch the effective potential and describe the possible orbits.

(d) Compute the proper time that it takes for a radial timelike geodesic starting at $r = r_0$ to reach $r = 0$. Similarly, compute the proper time that it takes for such a radial geodesic to reach $r = \infty$ from $r = 0$. For simplicity, consider geodesics with energy $E = 1$.