WEEK 10

Black holes

We will consider Schw. spacetime for all values of the radial coordinate r

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-2M/r} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

To understand the nature of black holes and the non-trivial causal structure of the spacetime we need to understand the light cones. To do this we consider radial (i.e., $\dot\theta = \dot\phi = 0$) null geodesics

$$0 = g_{ab}\dot{x}^a\dot{x}^b = -\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \frac{1}{1-\frac{2M}{r}}\left(\frac{dr}{d\lambda}\right)^2$$

$$\Rightarrow \quad \frac{dt}{dr} = \pm\frac{1}{1-\frac{2M}{r}}$$

This expression gives the slopes of the light cones in the (t,r) coordinates. For $r\to\infty$, $\frac{dt}{dr}\to\pm 1$ like in Minkowski space, but $\frac{dt}{dr}\to\pm\infty$ as $r\to 2M$. This suggests that according to observer far away, it would take infinite time for a light ray departing from $r=2M$ to reach them; similarly, it would seem to take infinite time for a light ray to reach $r=2M$. However, this is an illusion caused by the coordinate singularity at $r=2M$; an observer can fall towards smaller radii and cross $r=2M$, but far away observers would see the signals more and more slowly.

To see this, let's calculate the proper time for an radially infalling observer (i.e., timelike geod) to reach $r=0$ from some $r=r_0>2M$

$$\left(1-\frac{2M}{r}\right)\dot{t}=E$$

$$-1=g_{ab}\dot{x}^a\dot{x}^b=-\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2+\frac{1}{1-2M/r}\left(\frac{dr}{d\tau}\right)^2$$

$$=\frac{1}{1-\frac{2M}{r}}\left(-E^2+\left(\frac{dr}{d\tau}\right)^2\right)$$

$E=1$: choice of energy

$$\Rightarrow \quad \frac{dr}{d\tau}=\pm\sqrt{-1+\frac{2M}{r}+E^2}=\pm\sqrt{\frac{2M}{r}}$$

infalling

$$\Rightarrow \Delta\tau=\int_{\tau_0}^{\tau}d\tau=-\frac{1}{\sqrt{2M}}\int_{r_0}^{0}dr\,\sqrt{r}=\frac{2}{3\sqrt{2M}}\,r_0^{3/2}$$

Note that proper time is an invariant, i.e, independent of the coordinates.

Going back to the radial null geodesics, we can integrate the equation to find

$$t = \int dt = \pm \int \frac{dr}{1-2M/r} = \pm r_* + \text{const}$$

$$r_* = r + 2M \ln\left(\frac{r}{2M} - 1\right) \rightarrow \text{tortoise coordinate}$$

$$dr_* = \frac{dr}{1-2M/r} \Rightarrow dr = \left(1 - \frac{2M}{r}\right) dr_*$$

In terms of r_*, the Schw metric becomes:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{1-2M/r}\left[\left(1-\frac{2M}{r}\right) dr_*\right]^2 + r^2 d\Omega^2_{(2)}$$

$$= \left(1 - \frac{2M}{r}\right)\left(-dt^2 + dr_*^2\right) + r^2 d\Omega^2_{(2)}$$

where r should be regarded as a function of r_*. In these coordinates the light cones are $t = \pm r_* + \text{const}$, so longer close up, but the metric is still singular at $r = 2M$; in these coordinates the surface $r = 2M$ has been pushed to $r_* = -\infty$

To proceed we define coordinates adapted to radial null geodesics:

$v = t + r_*$ → infalling radial null geods: $v = \text{const}$

$u = t - r_*$ → outgoing radial null geods: $u = \text{const}$

Changing coordinates (ingoing Eddington-Finkelstein)

$$t = v - r_* \Rightarrow dt = dv - dr_* = dv - \frac{dr}{1-2M/r}$$

$$\Rightarrow ds^2 = -\left(1-\frac{2M}{r}\right)\left(dv - \frac{dr}{1-2M/r}\right)^2 + \frac{dr^2}{1-2M/r} + r^2 d\Omega^2_{(2)}$$

$$= -\left(1-\frac{2M}{r}\right) dv^2 + 2\, dv\, dr + r^2 d\Omega^2_{(2)}$$

→ this metric is smooth and invertible at $r = 2M$, which shows that $r = 2M$ is a mere coordinate singularity in the original Schw. metric.

· Radial null geodesics in ingoing EF coordinates:

$$0 = g_{ab}\dot{x}^a \dot{x}^b = -\left(1-\frac{2M}{r}\right)\dot{v}^2 + 2\dot{v}\dot{r}$$

$$= \dot{v}\left[-\left(1-\frac{2M}{r}\right)\dot{v} + 2\dot{r}\right]$$

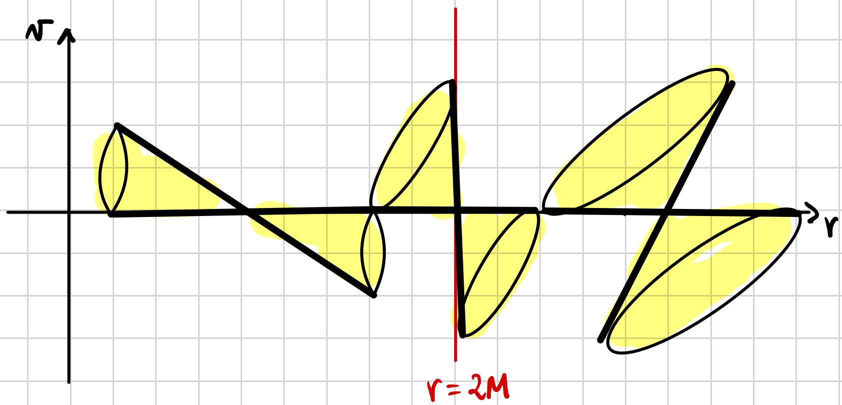
$\Rightarrow \quad \dot{v} = 0 \rightarrow v = \text{const}$

$$-\left(1-\frac{2M}{r}\right)\dot{v}+2\dot{r}=0 \Rightarrow \frac{dr}{dv}=\frac{\dot{r}}{\dot{v}}=\frac{2}{1-2M/r}$$

Far away, i.e., $r>2M$, $\frac{dr}{dv}>0 \rightarrow r$ advances as v advances so these are outgoing. Then $v=\text{const}$ are ingoing.

Note that in these coordinates the light cones are well-behaved:

- Along ingoing null geodesics, $v=\text{const}$, r varies from ∞ to 0, so it is possible to cross $r=2M$. On the surface, the light cones tilt inwards (i.e, towards smaller r) since $\frac{dr}{dv}<0$ for $r<2M$. The surface $r=2M$ is a point of no return: inside this surface, al future directed paths go towards smaller r.

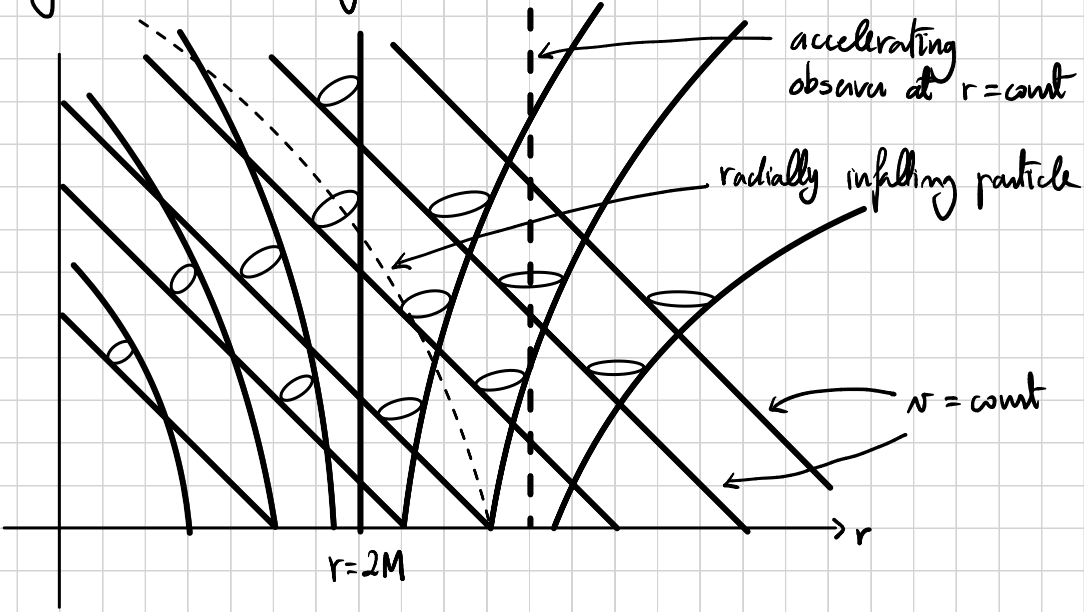


- $r = 2M$ is a point of no return: any observer (inertial or not) that dips below it can never escape. A surface past which particles can never escape to infinity is the event horizon of the black hole

- The event horizon is a null surface (i.e, the vectors tangent to it are null). Since nothing can escape the event horizon hence the name black hole. A black hole is simply a region of spacetime separated from infinity by an event horizon
- The notion of an event horizon is a global one. Locally, there is nothing special about the surface $r = 2M$

- Note that the interior of black holes ($r < 2M$) is essentially empty
- $r = 0$ is a genuine singularity: curvature and hence tidal forces become infinite!

· A Finkelstein diagram is a representation of the null geodesics in the (v, r) coordinates:



For a general static and spherically symmetric spacetime of the form

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2\,d\Omega^2_{(2)}$$

with $f(r)$ vanishing at some $r = r_+$ (i.e., $f(r_+) = 0$) represents a black hole. Ingoing/outgoing EF coordinates that are regular at $r = r_+$ can be found as follows:

ingoing: $dt = dv - \frac{dr}{f(r)}$

$$\Rightarrow \quad ds^2 = -f(r)\,dv^2 + 2\,dv\,dr + r^2 d\Omega^2_{(2)}$$

outgoing: $dt = du + \frac{dr}{f(r)}$

$$\Rightarrow \quad ds^2 = -f(r)\,du^2 - 2\,du\,dr + r^2 d\Omega^2_{(2)}$$

- <u>More general black holes</u>: Kerr's black hole
- Most astrophysical objects, e.g., stars, galaxies, etc., rotate so if black holes form in natural astrophysical processes then they should also rotate.
- a rotating black hole that is a solution of EVE was found in 1963 by Kerr:

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4Mar\sin^2\theta}{\Sigma}\,dt\,d\phi + \frac{\Sigma}{\Delta}\,dr^2$$

$$+ \Sigma\,d\theta^2 + \frac{\sin^2\theta}{\Sigma}\left[(r^2+a^2)^2 - a\,\Delta\sin^2\theta\right] d\phi^2$$

$\Delta = r^2 - 2Mr + a^2$, $\Sigma = r^2 + a^2\cos^2\theta$

M: mass

$a = J/M$: angular momentum/spin per unit mass.

- Event horizons occur at

$$\Delta = 0 \Rightarrow r_{\pm} = M \pm \sqrt{M^2 - a^2}$$

so we need $M > a$

- There is a physical singularity at $\Sigma = 0$

$\Rightarrow r = 0$ <u>and</u> $\theta = \frac{\pi}{2}$ $\rightarrow$ this is a ring.

- The Kerr spacetime is independent of t ($\rightarrow$ stationary) and ϕ ($\rightarrow$ axisymmetric) and hence it has two Killing vector fields: $K^a = (\partial_t)^a$ and $R^a = (\partial_\phi)^a$ but it is <u>not</u> static: $t \to -t$ is not a symmetry of ds^2. If we send $t \to -t$ we also need $\phi \to -\phi$ so that ds^2 is left invariant: if we go backwards in time we also have to reverse the sense of the rotation.

- Note that $K^a K_a = -\frac{1}{\Sigma}(\Delta - a^2 \sin^2\theta) = 0$ outside the horizon ($\Delta = 0$). The surface where $K^2 = 0$ is known as the ergosurface, and the region between the horizon and the ergosurface is the ergoregion. Inside the ergoregion, observers <u>must</u> move in the direction of rotation of the black hole

- The Killing vector field that is tangent to the null generators of the horizon is

$$\chi = \frac{\partial}{\partial t} + \Omega_H \frac{\partial}{\partial \phi} \quad , \quad \Omega_H = \frac{a}{r_+^2 + a^2}$$

Ω_H is the angular velocity of the black hole.

- <u>Uniqueness thms and astrophysical relevance of bhs.</u>
  - The Kerr black hole is believed to be stable under small perturbations
  - Stationary, asymptotically flat solutions to the EVE are uniquely characterised by their mass and spin and are given by the Kerr family of solutions

$\Rightarrow$ According to GR, all black holes in the Universe are given by the Kerr solution!