Example (Normal) $N(\mu, \sigma^2)$

$$f_Y(y; \theta, \phi) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)$$

$$= \exp\left(-\frac{1}{2}\ln(2\pi\sigma^2) - \frac{(y-\mu)^2}{2\sigma^2}\right)$$

$$= \exp\left(\frac{y\mu - \frac{\mu^2}{2}}{\sigma^2} - \frac{1}{2}\left(\frac{y^2}{2\sigma^2} + \ln(2\pi\sigma^2)\right)\right)$$

$\theta = \mu \quad \phi = \sigma^2, \quad b(\theta) = \frac{\theta^2}{2}, \quad \cancel{\phi} \; a(\phi) = \phi$

$$c(y, \phi) = -\frac{1}{2}\left(\frac{y^2}{2\phi} + \ln(2\pi\phi)\right)$$

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It can be shown for any $Y$ in the exponential family

$$E(Y) = b'(\theta)$$
$$Var(Y) = a(\phi)\, b''(\theta)$$

For normal $b'(\theta) = \cancel{2}\,\theta = \mu$ ✓

$$a(\phi) \cdot b''(\theta) = \phi \cdot 1 = \sigma^2$$ ✓

Example (Poisson)   Poisson($\lambda$)

$$f_Y(y;\theta,\phi) = \frac{e^{-\lambda}\lambda^{y}}{y!}$$

$$= \exp\left(-\lambda + y\ln\lambda - \ln y!\right)$$

$$= \exp\left(y\ln\lambda - \lambda - \ln y!\right)$$

$\theta = \ln\lambda$, $b(\theta) = \lambda = e^{\ln\lambda} = e^{\theta}$

Set $\phi = 1$, $a(\phi) = 1$   $c(y,\phi) = -\ln y!$

Example (Binomial)   $Z \sim \text{Binomial}(n,p)$

Set $Y = \frac{Z}{n}$, the proportion of trials which are successes

$$f_Z(z;\theta,\phi) = \binom{n}{z} p^{z}(1-p)^{n-z}$$

$Z = nY$

$$f_Y(y;\theta,\phi) = \binom{n}{ny} p^{ny}(1-p)^{n-ny}$$

$$= \exp\left(n\left\{y\ln p + (1-y)\ln(1-p)\right\} + \ln\binom{n}{ny}\right)$$

$$= \exp\left(n\left\{y\ln\frac{p}{1-p} + \ln(1-p)\right\} + \ln\binom{n}{ny}\right)$$

$\theta = \ln\frac{p}{1-p}$   $b(\theta) = -\ln(1-p) = -\ln\left(1 - \frac{e^{\theta}}{1+e^{\theta}}\right)$

because $e^\theta = \frac{p}{1-p} \Rightarrow e^\theta - pe^\theta = p \Rightarrow p = \frac{e^\theta}{1+e^\theta}$

$\phi = n \quad a(\phi) = 1/\phi, \quad c(y,\phi) = \ln\binom{n}{\phi y}.$

## Example (Gamma)

$\text{Gamma}(\alpha, \lambda)$

$$f_Y(y,\theta,\phi) = \frac{\lambda^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\lambda y}$$

$\mu = E(Y) = \frac{\alpha}{\lambda}$

↑

new parameter

$$= \exp\left(\alpha\ln\lambda - \ln\Gamma(\alpha) + (\alpha-1)\ln y - \lambda y\right)$$

~~Sset~~ Set

so $\lambda = \frac{\alpha}{\mu}$

$$= \exp\left(\alpha\ln\lambda - \ln\Gamma(\alpha) + (\alpha-1)\ln y - \frac{\alpha y}{\mu}\right)$$

new parameters are $\mu$ and $\alpha$

$$= \cancel{\exp\left(\left(-\frac{y}{\mu}\right.\right.} \exp\left(\alpha\ln\frac{\alpha}{\mu} - \ln\Gamma(\alpha) + (\alpha-1)\ln y - \frac{\alpha y}{\mu}\right)$$

$$= \exp\left[\left(-\frac{y}{\mu} - \ln\mu\right)\alpha + (\alpha-1)\ln y - \ln\Gamma(\alpha) + \alpha\ln\alpha\right]$$

$\theta = \frac{1}{\mu}, \; b(\theta) = \ln\left(-\frac{1}{\theta}\right)$

$\phi = \alpha, \; a(\phi) = \frac{1}{\phi}, \; c(y,\phi) = (\phi-1)\ln y + \phi\ln\phi - \ln\Gamma(\alpha)$

$\theta = -\frac{1}{\mu}$ but we can take $\theta = \frac{1}{\mu}$