MTH6127 **Metric Spaces and Topology**

Course work 12 **12 April, 2024**

1. Show that a finite union of compact subsets of X is compact.

   Assume that $X = \cup_{i=1}^{n} X_i$ where each space $X_1, \dots, X_n$ is compact. Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of X. Since X_i is compact, there is a finite subset $A_i \subset A$ such that $\cup_{\alpha\in A_i} U_\alpha \supset X_i$. Then $\{U_\alpha\}_{\alpha\in B}$ is a finite subcover of $\{U_\alpha\}_{\alpha\in A}$ where $B = \cup_{i=1}^{n} A_i$. Hence, X is compact.

2. Show that a discrete metric space is compact if and only if it is finite.

   Any finite topological space is compact - it is obvious from the definition.

   If X has discrete topology then every singleton $\{x\}$ is open. If X is a discrete topological space then the open cover $\mathcal{U} = \{\{x\}\}_{x\in X}$ has no proper subcovers and therefore if X is infinite then $\mathcal{U}$ has no finite subcovers. Thus, an infinite discrete topological space is not compact.

3. Apply the definition of compactness to show that the half-open interval $(0, 1]$ is not compact.

   The collection $\{(\frac{1}{n}, 1]\}_{n=1,2,\dots}$ is an open cover of $(0, 1]$ with no finite subcover.

4. Consider the map $f : [0, 1) \to S^1$ given by

   $$f(t) = (\cos 2\pi t, \sin 2\pi t).$$

   Here $S^1 \subset \mathbb{C}$ is the circle $S^1 = \{z \in \mathbb{C}; |z| = 1\}$.

   Show that f is continuous, bijective but is not a homeomorphism.

   The circle S^1 is compact (as a closed and bounded subset of $\mathbb{C} = \mathbb{R}^2$). The half-open interval $(0, 1]$ is not compact (as shown above). Hence no homeomorphism between $(0, 1]$ and S^1 exists.

5. Show that the spaces $[0, 1)$ and S^1 are not homeomorphic.

   See above.

6. Let $p : X \to Y$ be a continuous map with the property that there exists a continuous map $f : Y \to X$ such that $p \circ f$ equals the identity map of Y. Show that p is a quotient map.

   Let $U \subset Y$ be such that $p^{-1}(U) \subset X$ is open. Since $p \circ f = 1_Y$ and f is a continuous map we have

   $$U = (p \circ f)^{-1}(U) = f^{-1}(p^{-1}(U)) \subset X$$

   is open. Hence we see that any subset $U \subset Y$ with $p^{-1}(U) \subset X$ is open, and thus $p : X \to Y$ is a quotient map.

7. Define an equivalence relation on the plane $X = \mathbb{R}^2$ as follows:

$$(x_1, y_1) \sim (x_2, y_2), \quad \text{if} \quad x_1 + y_1^2 = x_2 + y_2^2.$$

Show that the quotient space $X^* = \mathbb{R}^2/\sim$ is homeomorphic to $\mathbb{R}$.

Define $p : \mathbb{R}^2 \to \mathbb{R}$ by $p(x, y) = x + y^2$. Define also $f : \mathbb{R} \to \mathbb{R}^2$ by $f(x) = (x, 0)$. Then p and f are continuous and $p \circ f(x) = x$ for any $x \in \mathbb{R}$. Thus by part (7), p is a quotient map, i.e. the quotient space $\mathbb{R}^2/\sim$ is homeomorphic to $\mathbb{R}$.

8. Define an equivalence relation on the plane $X = \mathbb{R}^2$ as follows:

$$(x_1, y_1) \sim (x_2, y_2), \quad \text{if} \quad x_1^2 + y_1^2 = x_2^2 + y_2^2.$$

Show that the quotient space $X^* = \mathbb{R}^2/\sim$ is homeomorphic to $[0, \infty)$.

Define $p : \mathbb{R}^2 \to [0, \infty)$ by $p(x, y) = x^2 + y^2$. Define also $f : [0, \infty) \to \mathbb{R}^2$ by $f(x) = (\sqrt{x}, 0)$. Then p and f are continuous and $p \circ f(x) = x$ for any $x \in [0, \infty)$. Thus by part (7), p is a quotient map, i.e. the quotient space $\mathbb{R}^2/\sim$ is homeomorphic to $[0, \infty)$.