MTH6127 **Metric Spaces and Topology**

Course work 10 **29 March 2024**

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1. Show that in the finite complement topology on $\mathbb{R}$ every subspace of $\mathbb{R}$ is compact.

   Let $\{U_i\}_{i\in J}$ be an open cover of $\mathbb{R}$. Any non-empty set U_{i_0} is of the form $\mathbb{R} - \{x_1, \dots, x_N\}$. Let U_{i_j} be an open set containing x_j, where $j = 1, \dots, N$. Then the sets $U_{i_0}, U_{i_1}, \dots, U_{i_N}$ form a finite subcover of the original cover. Hence $\mathbb{R}$ is compact when viewed with the finite complement topology.

2. Which of the following subsets of the real line $\mathbb{R}$ are compact; briefly explain your answer:

   (a) $(0, 1]$;

   Not compact since it is not closed.

   (b) $[0, 1)$;

   Not compact since it is not closed.

   (c) The Cantor set $F \subset [0, 1]$;

   Compact; it is a closed and bounded subset of $\mathbb{R}$.

   (d) $[0, \infty)$;

   Not compact since it is not bounded.

   (e) $\mathbb{R} - \{0\}$.

   Not compact since it is not closed and not bounded.

3. Let $f : X \to Y$ be a continuous map between metric spaces. Show that if X is compact then for any closed subset $F \subset X$ the image $f(F) \subset Y$ is closed.

   If $F \subset X$ is closed then F is compact and hence $f(F)$ is compact; then $f(F) \subset Y$ is closed since Y is Hausdorff.

4. For $p \in [1, \infty)$ consider the space ℓ_p of all infinite sequences $(x_1, x_2, \dots)$ of real numbers satisfying $\sum_{n\geq 1} |x_n|^p < \infty$.

   (a) Show that the unit ball $B[0; 1] \subset \ell_p$ is not compact.

   Consider the points $e_n \in \ell_p$ where $e_n = (0, \dots, 0, 1, 0, \dots)$ and 1 stands on the n-th place. We have an infinite sequence of points $e_n \in B[0; 1]$ and for $n \neq m$ the distance

$$d_p(e_n, e_m) = 2^{1/p}.$$

We see that this sequence has no convergent subsequences and hence $B[0;1]$ is not compact.

(b) Show that $B[0;1] \subset \ell_p$ is closed and bounded.

The complement $B[0;1]^c$ consists of the points $y \in \ell_p$ such that $d_p(0,y) > 1$. Then $B(y; (d_p(0,y) - 1)) \subset B[0;1]^c$, i.e. $B[0;1]$ is closed. The fact that $B[0;1]$ is bounded is obvious.

5. Let (X, d) be a metric space; let A be a non-empty subset of X. For each $x \in X$ define the distance from x to A by the equation

$$d(x, A) = \inf\{d(x, a); a \in A\}.$$

(a) Show that $x \mapsto d(x, A)$ is a continuous function of x.

Let $x, y \in X$ and $a \in A$. Then

$$d(x, a) \leq d(y, a) + d(x, y),$$

which implies by passing to the infimum for $a \in A$ that

$$d(x, A) \leq d(y, A) + d(x, y).$$

Similarly one shows that

$$d(y, A) \leq d(x, A) + d(x, y),$$

i.e.

$$|d(x, A) - d(y, A)| \leq d(x, y).$$

This implies that $d(x, A)$ is a continuous function of x.

(b) Show that if $A \subset X$ is compact then there exists $a \in A$ with $d(x, A) = d(x, a)$.

If A is compact then the continuous function $a \mapsto d(x, a)$ attains its infimum $d(x, A)$.

(c) Show that if $A \subset X$ is closed and $x \notin A$ then $d(x, A) > 0$.

If $A \subset X$ is closed and $x \notin A$ then there exists $\epsilon > 0$ such that $B(x, \epsilon) \cap A = \emptyset$. Hence for any $a \in A$ one has $d(x, a) \geq \epsilon$ and therefore $d(x, A) \geq \epsilon > 0$.

(d) Define the ϵ-neighbourhood of A in X to be the set

$$U(A, \epsilon) = \{x; d(x, A) < \epsilon\}.$$

Show that $U(A, \epsilon)$ equals the union of open balls $B(a, \epsilon)$ for $a \in A$.

It is obvious.

(e) Assume that $A \subset X$ is compact and $U \subset X$ is an open set containing A. Show that U contains some ϵ-neighbourhood of A.

Let $F = U^c$ be the complement of U. F and A are disjoint closed sets. For $a \in A$ the function $a \mapsto d(a, F)$ is a continuous function of $a \in A$ (by part (a)). Since A is compact this function attains its infimum, i.e. there exists $a_0 \in A$ such that $d(a_0, F) = \inf_{a \in A} d(a, F)$. Since F is closed and $a_0 \notin F$ we have $d(a_0, F) = \inf_{a \in A} d(a, F) = \epsilon > 0$, by part (c). Then clearly the ϵ-neighbourhood of A is contained in U.

(f) Is the previous statement true if $A \subset X$ is closed but non-compact?

The previous statement can be false if $A \subset X$ is closed but non-compact. Consider the following example. Let $X = \mathbb{R}^2$, $A = \{(x, 0); x \geq 0\}$ and $U = \{(x, y); |xy| < 1\}$. U is open and contains A but no ϵ-neighbourhood of A is contained in U.