MTH6127 **Metric Spaces and Topology**

Course work 9 **22 March 2023**

1. Let X and Y be topological spaces and let $f : X \to Y$ be a map. Show that the following properties of f are equivalent:
   (a) f is continuous;
   (b) For every subset $A \subset X$ one has $f(\overline{A}) \subset \overline{f(A)}$.
   (c) For every closed set $F \subset Y$ the preimage $f^{-1}(F) \subset X$ is closed in X.

(a) $\implies$ (b). For $x \in \overline{A}$ consider an open subset $f(x) \in U \subset Y$. Then $f^{-1}(U) \subset X$ is open and contains x. Therefore $f^{-1}(U) \cap A \neq \emptyset$ and hence $U \cap f(A) \neq \emptyset$. This shows that $f(x) \in \overline{f(A)}$, i.e. $f(\overline{A}) \subset \overline{f(A)}$.

(b) $\implies$ (c). Let $F \subset Y$ be closed. Denote $A = f^{-1}(F) \subset X$. Then $f(\overline{A}) \subset \overline{f(A)} = \overline{F} = F$ which shows that $\overline{A} \subset f^{-1}(F) = A$, i.e. A is closed.

(c) $\implies$ (a). This was explained in lectures.

2. In the finite complement topology on $\mathbb{R}$, to what point (or points) does the sequence $x_n = 1/n$ converge?

This sequence converges to any $x_0 \in \mathbb{R}$ as any neighbourhood of x_0 contains all terms of the sequence x_n except finitely many.

3. Let $y_n = 1$ for n even and $y_n = -1$ for n odd. In the finite complement topology on $\mathbb{R}$, to what point (or points) does the sequence y_n converge?

   This sequence has no limit.

4. Show that a subspace of a Hausdorff space is Hausdorff.

Let X be Hausdorff and $A \subset X$. For $x, y \in A$, $x \neq y$ we may find open $x \in U \subset X$ and $y \in V \subset X$ with $U \cap V = \emptyset$. Then $U' = U \cap A$ and $V' = V \cap A$ are open disjoint subsets of A containing x and y correspondingly, i.e. A is Hausdorff.

5. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be defined by the equation

$$F(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & \text{if} \quad (x,y) \neq (0,0), \\ 0, & \text{if} \quad (x,y) = (0,0). \end{cases}$$

   (a) Show that for any $x_0 \in \mathbb{R}$ the function $y \mapsto F(x_0, y)$ is continuous.
   If $x_0 \neq 0$ the $F(x_0, y)$ is continuous as ratio or two polynomials with nonzero denominator. If $x_0 = 0$ then $F(x_0, y) = 0$ for any y.

   (b) Show that for any $y_0 \in \mathbb{R}$ the function $x \mapsto F(x, y_0)$ is continuous.
   As above.

   (c) Show that the function $x \mapsto F(x, x)$ is discontinuous.
   For $x \neq 0$ we have $F(x, x) = 1/2$ and for $x = 0$, we have $F(x, x) = 0$.

   (d) Show that $F : \mathbb{R}^2 \to \mathbb{R}$ is discontinuous.
   F is discontinuous since its restriction onto the line $x = y$ is discontinuous.

6. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is not continuous at every point $x \in \mathbb{R}$.

   Let $f(x) = 1$ for all $x \in \mathbb{R}$ rational and $f(x) = -1$ for all $x \in \mathbb{R}$ irrational. Then f is discontinuous at every $x \in \mathbb{R}$ as it can be represented as the limit of either rational or irrational numbers.

7. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous at a single point $x_0 \in \mathbb{R}$.

   Let $f(x) = x - x_0$ for all $x \in \mathbb{R}$ rational and $f(x) = x_0 - x$ for all $x \in \mathbb{R}$ irrational. Then f is discontinuous at every $x \in \mathbb{R} - \{x_0\}$ and it is continuous at $x = x_0$.