3^{rd} year pathway presentation at 17:00

Recap quiz

1) What is the dual of

$$\begin{aligned} \max \quad & \underline{c}^T \underline{x} \\ \text{sub to} \quad & A\underline{x} \leq \underline{b} \\ & \underline{x} \geq \underline{0} \end{aligned}$$

$$\begin{aligned} \min \quad & \underline{b}^T \underline{y} \\ \text{sub to} \quad & A^T \underline{y} \geq \underline{c} \\ & \underline{y} \geq \underline{0} \end{aligned}$$

2)

	Primal	Dual
Goal	max	min
# Variables	n	m
# unrestricted variables	n'	m'
# constraints	m	n
# equality constraints	m'	n'

3) Weak duality theorem says:

If $\underline{x}$ is a feasible to LP above
and $\underline{y}$ is a feasible to its dual

then $\underline{b}^T\underline{y} \geq \underline{c}^T\underline{x}$

Strong duality theorem says:

If $\underline{x}$ is a optimal to LP above
and $\underline{y}$ is a optimal to its dual

then $\underline{b}^T\underline{y} = \underline{c}^T\underline{x}$

Application of duality (Supervisor problem)

Your supervisor asks you to solve a very large LP

$$\max \ \underline{c}^T\underline{x}$$
$$\text{sub to } A\underline{x} \le \underline{b}$$
$$\underline{x} \geq \underline{0}$$

You find an optimal solution $\underline{x}^*$ e.g. using simplex.

How can you *quickly* convince your supervisor that $\underline{x}^*$ is optimal (supervisor doesn't want to check all your simplex steps)

Ans: Give your supervisor both $\underline{x}^*$ and the optimal solution $\underline{y}^*$ to the dual.

What three quick checks must superviser do to be convinced that $\underline{x}^*$ is optimal?

Supervisor checks (a) $\underline{x}^*$ is feasible for primal LP

(b) $\underline{y}^*$ is feasible for dual LP

(c) $\underline{c}^T\underline{x}^* = \underline{b}^T\underline{y}^*$

If so, supervisor concludes $\underline{x}^*$ is optimal for primal and $\underline{y}^*$ is optimal for dual why?

$\underline{y}^*$ sometimes called certificate of optimality.

Why can supervisor conclude this?

Suppose $\underline{x}^*$ is not optimal.

So $\exists$ feasible solution $\underline{x}$ s.t. $\underline{c}^T\underline{x} > \underline{c}^T\underline{x}^* \overset{(c)}{=} \underline{b}^T\underline{y}^*$

But $\underline{c}^T\underline{x} > \underline{b}^T\underline{y}^*$ contradicts weak duality thm,

So $\underline{x}^*$ optimal.

Complementary slackness

Thm (Principle of complementary slackness)

Suppose we have an LP in standard inequality form

$$\max \underline{c}^T \underline{x} \quad \text{sub to } A\underline{x} \leq \underline{b}, \quad \underline{x} \geq \underline{0}$$

with n variables, m constraints

i^{th} constraint in primal says $(A\underline{x})_i \leq b_i$

$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i$

So its dual is

$$\min \underline{b}^T \underline{y} \quad \text{sub to } A^T\underline{y} \geq \underline{c}, \quad \underline{y} \geq \underline{0}$$

has m variables, n constraints

j^{th} constraint of dual says $(A^T\underline{y})_j \geq c_j$

$a_{1j}y_1 + a_{2j}y_2 + \cdots + a_{mj}y_m \geq c_j$

We have $\underline{x}$ is optimal for the primal LP and $\underline{y}$ is optimal for the dual if and only if

(a) $\underline{x}$ is feasible for primal LP and $\underline{y}$ is feasible for dual LP

(b) for each j we have $x_j = 0$ or $(A^T\underline{y})_j = c_j$

(so j^{th} variable of primal LP is 0 or j^{th} constraint of dual is tight)

(c) for each i we have $y_i = 0$ or $(A\underline{x})_i = b_i$

(so i^{th} variable of dual is 0 or i^{th} constraint of primal LP is tight)

Pf omitted this year and non-examinable

Example of using complementary slackness

Example 9.1. Consider the linear program:

$$\begin{array}{lrl} \text{maximise} & 2x_1 - x_2 + 8x_3 & \\ \text{subject to} & 2x_3 \le 1 & \text{con 1} \\ & 2x_1 - 4x_2 + 6x_3 \le 3 & \text{con 2} \\ & -x_1 + 3x_2 + 4x_3 \le 2 & \text{con 3} \\ & x_1, x_2, x_3 \ge 0 & \end{array} \qquad (9.5)$$

Show that $x_1 = 17/2$, $x_2 = 7/2$, $x_3 = 0$ is an optimal solution to this program.

Idea: check whether given $\underline{x}$ and some (to be determined) $\underline{y}$ satisfy (a), (b), (c) in complementary slackness and if so, can conclude $\underline{x}$ is optimal for LP above and $\underline{y}$ is optimal for its dual.

1) write down the dual:

$$\begin{array}{lrl} \text{minimise} & y_1 + 3y_2 + 2y_3 & \\ \text{sub to} & 2y_2 - y_3 \geq 2 & \text{con 1}' \\ & -4y_2 + 3y_3 \geq -1 & \text{con 2}' \\ & 2y_1 + 6y_2 + 4y_3 \geq 8 & \text{con 3}' \\ & y_1, y_2, y_3 \geq 0. & \end{array}$$

2) check given solution $\underline{x}$ is feasible
which primal constraints are tight?
Which primal variables are zero?

con 1	$2 \times 0 \le 1$ ✓	not tight	$x_1 \neq 0$
con 2	$2 \times \frac{17}{2} - 4 \times \frac{7}{2} + 6 \times 0 = 3$ ✓	tight	$x_2 \neq 0$
con 3	$-\frac{17}{2} + 3 \times \frac{7}{2} + 4 \times 0 = 2$ ✓	tight	$x_3 = 0$

3) Complementary slackness (b), (c) gives us equations that $\underline{y}$ must satisfy if $\underline{x}$ is optimal for primal LP and $\underline{y}$ is optimal for dual

$y_1 = 0$

con 1' and con 2' tight

$$2y_2 - y_3 = 2$$
$$-4y_2 + 3y_3 = -1$$

3) Complementary slackness (b), (c) gives us equations that $\underline{y}$ must satisfy if $\underline{x}$ is optimal.

$y_1 = 0$ con 1' and con 2' tight

$$2y_2 - y_3 = 2 \quad (1)$$
$$-4y_2 + 3y_3 = -1 \quad (2)$$

4) solve equations to determine $\underline{y}$

$y_1 = 0$ $2(1) + (2)$ $y_3 = 3$

$y_2 = 5/2$

By our choice of $\underline{y}$, $\underline{x}$ and $\underline{y}$ satisfy (b) and (c) from the theorem.

5) Check $\underline{y}$ is feasible

con 1', con 2' ✓ holds by construction

con 3' $2y_1 + 6y_2 + 4y_3 \geq 8$

$$2\times 0 + 6\times \tfrac{5}{2} + 4\times 3 = 27 \geq 8 \checkmark$$

6) Conclusion: have found that $\underline{x}$, $\underline{y}$ satisfy complementary slackness (i.e. (a), (b), (c) from theorem), hence $\underline{x}$ is optimal for primal
$\underline{y}$ is optimal for dual

How would we show $\underline{x}$ is not optimal?

- If there's no solution for $\underline{y}$ in step 4 then $\underline{x}$ is not optimal (by theorem)
- If $\underline{y}$ is not feasible in step 6 then $\underline{x}$ is not optimal (by theorem).

When solving equations in step 4 if the solution for $\underline{y}$ is not unique check if any of them is feasible for the dual. If so then $\underline{x}$ is optimal for primal LP
If not then $\underline{x}$ is not optimal for primal LP.

Modelling Revisited (non-linear objective)

Example 9.2. A factory makes 2 different parts (say, part X and part Y). Their plant has 4 separate processes in place: there are two older processes (say, process 1 and 2) that produce parts X and Y directly, as well as two different integrated processes for producing both X and Y simultaneously. The 4 processes can be run simultaneously, but require labour, raw metal, and electricity. The hourly inputs and outputs for each process are as follows:

	Outputs		Inputs		
Process	X	Y	Metal	Electricity	Labour
1	4	0	100 kg	800 kWh	16 hrs
2	0	1	70 kg	600 kWh	16 hrs
3	3	1	120 kg	2000 kWh	50 hrs
4	6	3	270 kg	4000 kWh	48 hrs

We have unlimited resources and wish to produce 120 parts X and 50 parts of Y
How quickly can this be done. (Processes can run simultaneously)

Ans: decision variables P_i = # hours we run process i $i=1,\dots,4$

dummy variables x, y = amount of X, Y produced

m, e, l = amount of metal, electricity and labour used

t = total time t.

minimise t

subject to
$$x = 4P_1 + 3P_3 + 6P_4$$
$$y = P_2 + P_3 + 3P_4$$
$$x \geq 120$$
$$y \geq 50$$
$$t = \max(P_1, P_2, P_3, P_4)$$
$$P_1, P_2, P_3, P_4 \geq 0, \quad x, y, t \text{ unrestricted.}$$

(not a linear program!)

(1) minimise t

subject to $x = 4P_1 + 3P_3 + 6P_4$

$y = P_2 + P_3 + 3P_4$

$x \geq 120$

$y \geq 50$

$t = \max(P_1, P_2, P_3, P_4)$

$P_1, P_2, P_3, P_4 \geq 0$, x, y, t unrestricted.

The following LP has same optimal solution as (1)

(2) minimise t

sub to $x = 4P_1 + 3P_3 + 6P_4$

$y = P_2 + P_3 + 3P_4$

$x \geq 120$

$y \geq 50$

$t \geq P_1$

$t \geq P_2$

$t \geq P_3$

$t \geq P_4$

$P_1, P_2, P_3, P_4 \geq 0$, x, y, t unrestricted

Why? True or false?

Every feasible solution of (1) is feasible for (2)

True: if $t = \max_{i=1,\dots,4}(P_i)$ then $t \geq P_i$ $i = 1, \dots, 4$.

Every feasible solution of (2) is feasible for (1)

False: if $t \geq P_i$ $\forall i$ could have $t > \max(P_i)$

Optimal solution of (2) is feasible for (1)

True: suppose not. If t^* is optimal for (2) but not feasible for (1) then this is because $t^* > \max(P_i)$. Then could reduce t^* while still satisfying all constraints in (2), a contradiction.

So optimal solution for (2) also optimal for (1).

This works generally when we have a program of following form (with variables $x_1, \ldots, x_k, y_1, \ldots, y_l$

$$\begin{aligned} &\text{minimise} && \left(\max(x_1, x_2, \ldots, x_k) \right) \\ &\text{sub to} && \text{linear constraints in variables} \\ &&& \text{sign restrictions of variables} \end{aligned}$$

This program has same optimal solution as LP below

minimise $\quad m$

sub to $\quad$ same linear constraints and

$m \geq x_1$
$m \geq x_2$
$\vdots$
$m \geq x_k$

same sign restrictions
m unrestricted.

<u>Remark</u> Same idea works for maximising a minimum but not for maximising a maximum or minimising a minimum (what goes wrong if we try to use the same method).

Piecewise linear concave/convex objectives

Example 9.2. A factory makes 2 different parts (say, part X and part Y). Their plant has 4 separate processes in place: there are two older processes (say, process 1 and 2) that produce parts X and Y directly, as well as two different integrated processes for producing both X and Y simultaneously. The 4 processes can be run simultaneously, but require labour, raw metal, and electricity. The hourly inputs and outputs for each process are as follows:

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4	6	3	270 kg	4000 kWh	48 hrs

limited resources:
6000 kg metal
100000 kWh electric
1000 hours labour

Revenue
Y sells for £1800 per unit
X sells for £1000 per unit

As before P_i = # hours of process i
m, e, l = amount of metal/electric/labour used
x, y = amount of X, Y produced.

$$
\begin{aligned}
\text{maximise} \quad & 1000x + 1800y \\
\text{subject to} \quad & x = 4p_1 + 3p_3 + 6p_4 \\
& y = p_2 + p_3 + 3p_4 \\
& m = 100p_1 + 70p_2 + 120p_3 + 270p_4 \\
& e = 800p_1 + 600p_2 + 2000p_3 + 4000p_4 \\
& l = 16p_1 + 16p_2 + 50p_3 + 48p_4 \\
& m \leq 6000 \\
& e \leq 100000 \\
& l \leq 1000 \\
& p_1, p_2, p_3, p_4 \geq 0 \\
& x, y \text{ unrestricted} \\
& m, e, l \text{ unrestricted}
\end{aligned}
$$

Piecewise linear concave/convex objectives

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4	6	3	270 kg	4000 kWh	48 hrs

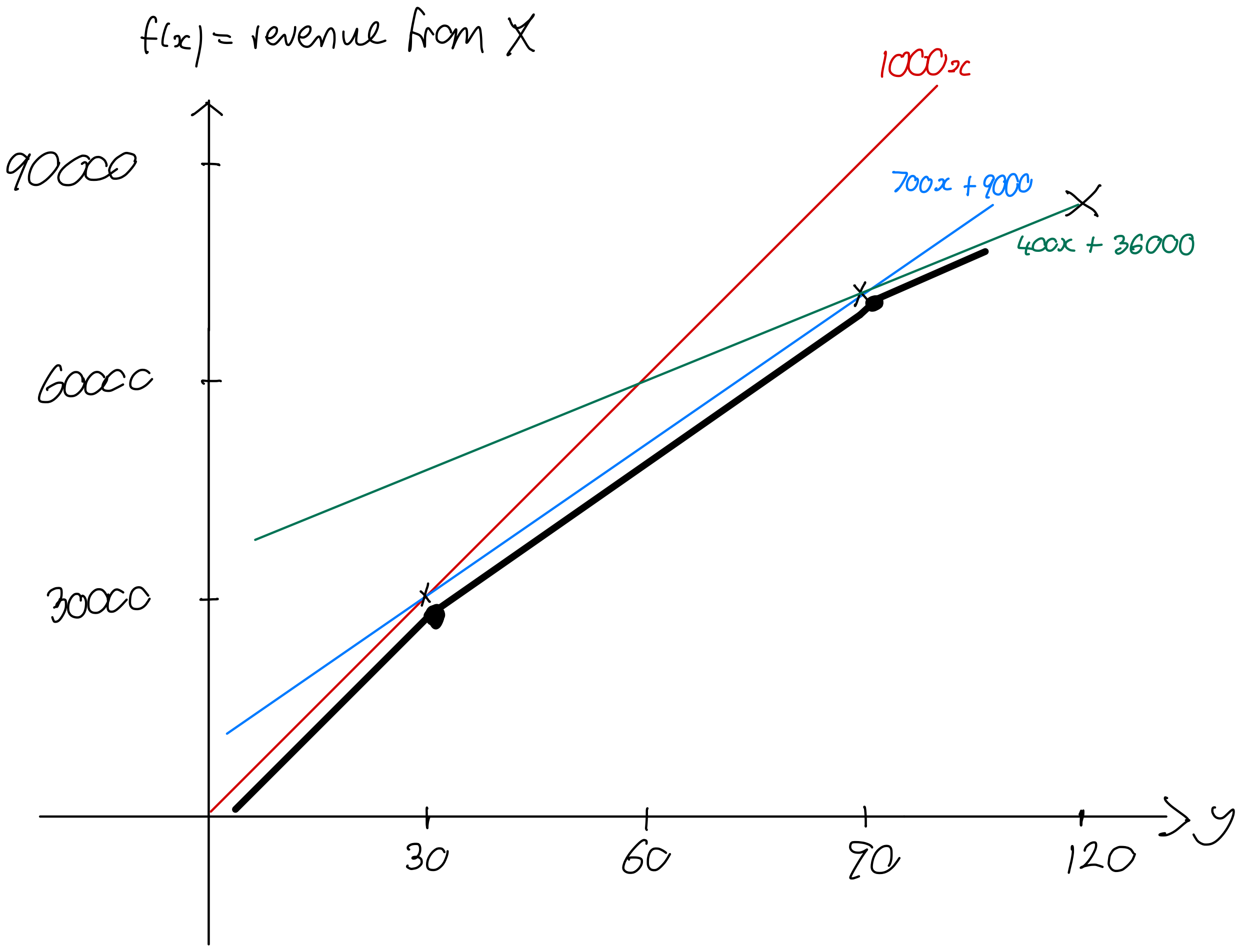
Limited resources:
6000 kg metal
100 000 kWh electric
1000 hours labour

Revenue
Y sells for £1800 per unit
X sells for £1000 per unit first 30
£700 next 60
£400 remaining

Revenue from Y = $1800y$
Revenue from X = $f(x)$
Idea: write $f(x)$ as a minimum.

$$f(x) = \begin{cases} 1000x & \text{if } x \in [0,30) \\ 1000 \times 30 + 700(x-30) & \text{if } x \in [30,90) \\ 1000 \times 30 + 700 \times 60 + 400(x-90) & \text{if } x \geq 90 \end{cases}$$

$$= \begin{cases} 1000x & \text{if } x \in [0,30) \\ 700x + 9000 & \text{if } x \in [30,90) \\ 400x + 36000 & \text{if } x \geq 90 \end{cases}$$



$$f(x) = \begin{cases} 1000x & \text{if } x \in [0,30) \\ 700x + 9000 & \text{if } x \in [30,90) \\ 400x + 36000 & \text{if } x \geq 90 \end{cases}$$

$$f(x) = \min(1000x,\ 700x + 9000,\ 400x + 36000)$$

$$
\begin{array}{lrl}
\text{maximise} & \overset{z}{\cancel{1000x}} + 1800y & \\
\text{subject to} & x = 4p_1 + 3p_3 + 6p_4 & \\
 & y = p_2 + p_3 + 3p_4 & \\
 & m = 100p_1 + 70p_2 + 120p_3 + 270p_4 & \\
 & e = 800p_1 + 600p_2 + 2000p_3 + 4000p_4 & \\
 & l = 16p_1 + 16p_2 + 50p_3 + 48p_4 & \\
 & m \leq 6000 & z \leq 1000x \\
 & e \leq 100000 & z \leq 700x + 9000 \\
 & l \leq 1000 & z \leq 400x + 36000 \\
 & p_1, p_2, p_3, p_4 \geq 0 & \\
 & x, y \text{ unrestricted} & \\
 & m, e, l \text{ unrestricted} & z \text{ unrestricted}
\end{array}
$$

This works because the slopes of the lines are decreasing as x increases (piecewise linear concave function)

Similar method works for maximising piecewise linear convex function.