**MTH6127** **Metric Spaces and Topology**

**Course work 8** **15 March 2024**

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1. Let $X$ be a topological space; let $A$ be a subspace of $X$. Suppose that for each $x \in A$ there is an open set $U$ containing $x$ such that $U \subset A$. Show that $A$ is open in $X$.

   For $x \in A$ let $U_x \subset A$ denote an open subset containing $x$. Then $A = \cup_{x \in A} U_x$ is open as a union of open subsets.

2. Let $Y$ be a subspace of $X$. If $U$ is open in $Y$ and $Y$ is open in $X$ then $U$ is open in $X$.

   "$U$ is open in $Y$" means that $U = V \cap Y$ where $V \subset X$ is open in $X$. If $Y$ is open in $X$ the the intersection $V \cap Y$ is open in $X$ (as intersection of two open subsets). Hence, $U = V \cap Y$ is open in $X$.

3. Let $Y$ be the subset $[0, 1) \cup \{2\}$ of $\mathbb{R}$. Show that in the subspace topology on $Y$ the single point $\{2\}$ is closed and open. Besides, show that the set $[0, 1)$ is closed and open in $Y$.

   The intersection $(1.5, 2, 5) \cap Y$ is $\{2\}$ and $[1.5, 2, 5] \cap Y = \{2\}$. This shows that $\{2\}$ is open and closed in $Y$. Its complement $[0, 1)$ is also open and closed in $Y$.

4. Show that if $Y$ is a subspace of $X$ and $A$ is a subspace of $Y$, then the topology $A$ inherits as a subspace of $Y$ is the same as the topology it inherits as a subspace of $X$.

   Denote by $\mathcal{T}$ the topology on $A$ induced by the topology of $X$; denote by $\mathcal{T}'$ the topology on $A$ induced by the topology of $Y$. Then $\mathcal{T}$ consists of the sets $U \cap A$ where $U \subset X$ is open and $\mathcal{T}'$ consists of the sets $U' \cap A$ where $U' \subset Y$ is open. Each such $U'$ has the form $U' = U \cap Y$ where $U \subset X$ is open. Then

   $$U' \cap A = (U \cap Y) \cap A = U \cap (Y \cap A) = U \cap A.$$

   Thus we see that $\mathcal{T} = \mathcal{T}'$.

5. Show that the set $A = \{(x, y); x \geq 0, \ y \geq 0\} \subset \mathbb{R}^2$ is closed.

   If $(x, y) \notin A$ then either $x < 0$ or $y < 0$. The open ball $B((x, y); r)$ is disjoint from $A$ when $0 < r < \min\{|x|, |y|\}$. Thus the complement $A^c$ is open and hence $A$ is closed.

6. In the finite complement topology on a set $X$, the closed sets consist of $X$ itself and all finite subsets of $X$.

   It is obvious.

7. Consider the following subset $Y = [0,1] \cup (2,3)$ of the real line $\mathbb{R}$. Show that both sets $[0,1]$ and $(2,3)$ are open and closed in the subspace topology of $Y$.

   This is similar to question (3).