MTH 4104 Example Sheet IV Solutions

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IV-1. $X^5 = (X^2 + (1 + \sqrt{-1})X + \sqrt{-1})(X^3 + (-1 - \sqrt{-1})X^2 + \sqrt{-1}X) + X$.

IV-2. $X^2 - X \in \mathbb{Z}_6[X]$ has $[0], [1], [3], [4]$ as solutions in $\mathbb{Z}_6$.

IV-3. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix} = \begin{pmatrix} 3a-2b & 4a-3b \\ 3c-2d & 4c-3d \end{pmatrix}$. On the other hand, $\begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 3a+4c & 3b+4d \\ -2a-3c & -2b-3d \end{pmatrix}$. Equating all entries, we get $a = -3c+d$ and $b = -2c$ while c and d are arbitrary elements of $\mathbb{R}$. The matrices that commute with $\begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix}$ are $\left\{ \begin{pmatrix} -3c+d & -2c \\ c & d \end{pmatrix} \mid c, d \in \mathbb{R} \right\}$.

IV-4. $\mathrm{M}_n(R[X])$ is the set of n-by$n$ matrices with entires polynomials in $R[X]$, while $(\mathrm{M}_n(R))[X]$ is the set of polynomials with coefficients in n-by-n matrices in $\mathrm{M}_n(R)$. For example, $\begin{pmatrix} 1 & X-1 \\ -X & 1 \end{pmatrix}$ is an element of $\mathrm{M}_2(\mathbb{Z}[X])$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is an element of $\mathrm{M}_2(\mathbb{Z})[X]$.

IV-5. Check $(AB)C = A(BC)$ by brute force.

IV-6. (R+0) $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} + \begin{pmatrix} p & q \\ 0 & r \end{pmatrix} = \begin{pmatrix} a+p & b+q \\ 0 & d+r \end{pmatrix}$. (R+1) Clear. (R+2) $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$. (R+3) The inverse of $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ is $\begin{pmatrix} -a & -b \\ 0 & -c \end{pmatrix}$. (R+4) Clear. (R×0) $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} p & q \\ 0 & r \end{pmatrix} = \begin{pmatrix} ap & aq+br \\ 0 & cr \end{pmatrix}$. (R×1) Clear. (R×+) Clear. (R+×) Clear.

IV-7. (a) There are three kinds of symmetries: the identity $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$, rotations with respect to the centre by $2\pi/5, 4\pi/5, 6\pi/5, 8\pi/5$ (for example, the $4\pi/5$ rotation is $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix}$) and reflections with to a line passing through a vertex and the midpoint of the opposite edge (for example, the reflection with respect to the vertex 1 is $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 4 & 3 & 2 \end{pmatrix}$). The following are all:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 4 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 4 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 2 & 1 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix}$$

(b) Yes.

IV-8. (a) $A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ are permutation matrices. Then

$AB = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is again a permutation matrix. In fact, the set of all permutation matrices in $\mathrm{M}_n(\mathbb{R})$ define a group under multiplication (the identity matrix *is* the identity element) (b) The permutation matrices are in bijection with the permutations of $\{1, \ldots, n\}$. Given a permutation f in S_n, define the matrix $M(f)$ by letting its (r, s)-entry to be 1 if $r = f(s)$ and 0 otherwise. For example, if $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$, then $M(f) = A$ above. The map $f \mapsto M(f)$ is a group homomorphism, in the sense that $M(f \circ g) = M(f)M(g)$, where composition in S_n is 'translated' into multiplication of permutation matrices in $\mathrm{M}_n(\mathbb{R})$.

If we define a (permutation) matrix $N(f)$ associated to f by letting its (r, s)-entry ro be 1 if $f(r) = s$ and 0 otherwise, then $N(f)$ turns out to be the transpose of $M(f)$.