RELATIVITY – MTH6132

PROBLEM SET 9

1. Riemannian manifolds for which the Ricci tensor is proportional to the metric tensor, $R_{ab} = \lambda g_{ab}$ are called *Einstein manifolds*.

(a) Show that in the case of of an n-dimensional Einstein manifold, we necessarily have $\lambda = R/n$, that is,

$$R_{ab} = \frac{R}{n} g_{ab}.$$

(b) Use the Bianchi identity to prove that for dimensions $n \geq 3$, Einstein manifolds necessarily have constant scalar curvature.

2. Let $(\mathcal{M}, g)$ be a manifold with metric.

(a) The *Laplace operator* on $\mathcal{M}$ is defiend by

$$\triangle f = \operatorname{div}(\operatorname{grad}(f)) = \nabla^a \nabla_a f\,.$$

Show that this is equivalent to

$$\triangle f = \frac{1}{\sqrt{|g|}} \sum_{i,j} \partial_i \left(g^{ij} \sqrt{|g|} \partial_j f \right),$$

where $|g| = \det g$ if g is Riemannian and $-\det g$ if g is Lorentzian.

(b) Let f be a spherically symmetric function on $\mathbb{R}^n$, i.e., let f be a function on $\mathbb{R}^n$ which depends only on $r = |x|$. Show that the Euclidean Laplacian satisfies

$$\triangle f = \frac{\partial^2 f}{\partial r^2} + \frac{n-1}{r} \frac{\partial f}{\partial r}.$$

Use this to find all spherically symmetric solutions of Laplace's Equation

$$\triangle f = 0\,.$$

3. Let $V = V^a$ be a Killing vector field, i.e.,

$$\nabla_{(a} V_{b)} = 0\,.$$

Show that

$$\nabla_a \nabla_b V^c = R^c{}_{bad}\, V^d\,.$$

4. Consider the general static spherically symmetric spacetime in four dimensions:

$$ds^2 = -e^{2A(r)} dt^2 + e^{2B(r)} dr^2 + r^2 (d\theta^2 + \sin^2\theta\, d\phi^2)$$

Compute the Christoffel symbols.

5. Consider the metric of **Problem 4** above. Compute:

(a) The components of the Riemann tensor.

(b) The components of the Ricci tensor.

6. This problem has two parts:

(a) Show that in the presence of a negative cosmological constant $\Lambda = -\frac{3}{L^2}$, the Einstein equations

$$G_{ab} + \Lambda g_{ab} = 0\,,$$

reduce to

$$R_{ab} + \frac{3}{L^2}\, g_{ab} = 0\,.$$

(b) Using the spherically symmetric ansatz for the metric as in **Problem 4** and the results in **Problem 5**, solve the Einstein equations with negative cosmological constant that you have derived in Part **(a)** to find the *Schwarzschild-anti de Sitter* spacetime.

7. Consider the Schwarzschild metric, given in local coordinates (t, r, θ, ϕ) by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

where we have set $G = 1$ (together with $c = 1$, these are called geometric units).

(a) Using any method, verify that the *Kretschman scalar* K is given by

$$K = R_{abcd}R^{abcd} = \frac{48M^2}{r^6}.$$

What does this calculation demonstrate?

(b) Introduce a new coordinate r_*, called the *tortoise coordinate*, as

$$r_* = r + 2M \ln\left(\frac{r}{2M} - 1\right).$$

Defining the ingoing null geodesics (light rays) by $v = t + r_*$ and the outgoing null geodesics by $u = t - r_*$, rewrite the Schwarzschild metric in Eddington-Finkelstein form as

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right) \mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2).$$

(c) Using the Eddington-Finkelstein form above, consider radial light rays ($d\theta = d\phi = 0$ and $ds^2 = 0$) to obtain

$$\frac{dv}{dr} = \frac{2}{1 - 2M/r}.$$

Integrate to express v as a function of r. This describes the paths followed by radial light rays in the (r, v) coordinates. Describe the behaviour of light cones by graphing light cones in the (r, v) plane. You will need to consider two regions: $r > 2M$ and $r < 2M$.

8. Consider the line element

$$ds^2 = \left(1 - \frac{2GM}{r} + \frac{GP^2}{r^2}\right) dt^2 + \left(1 - \frac{2GM}{r} + \frac{GP^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,,$$

where M and P are constants satisfying $M > P > 0$.

1. Find the expressions for the conserved energy E and angular momentum L along the geodesics associated to the Killing vectors $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \phi}$ respectively.

2. Find the effective potential $V(r)$ that governs the radial motion of the geodesics:
$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V(r) = \mathcal{E}\,,$$
where λ is the affine parameter along the geodesics and $\mathcal{E} = \frac{1}{2}E^2$ is defined in the lecture notes.

3. Sketch $V(r)$ for the null geodesics. You should assume $L > 0$.

4. Does this spacetime have any horizon(s)? If so, write down the spacetime metric above in ingoing Eddington-Finkelstein coordinates that are smooth at the horizon(s).

9. Consider the following spacetime,

$$ds^2 = -\left(\frac{r^2}{\ell^2} - \frac{r_0^2}{r^2}\right) dt^2 + \left(\frac{r^2}{\ell^2} - \frac{r_0^2}{r^2}\right)^{-1} dr^2 + r^2(dx^2 + dy^2 + dz^2)\,, \qquad (1)$$

where ℓ and r_0 are real and positive constants.

1. Show that the conserved quantities along the geodesics are given by:
$$E = \left(\frac{r^2}{\ell^2} - \frac{r_0^2}{r^2}\right) \dot{t}\,, \quad k_x = r^2\,\dot{x}\,, \quad k_y = r^2\,\dot{y}\,, \quad k_z = r^2\,\dot{z}\,,$$
where $\dot{x} = \frac{dx}{d\tau}$, etc., and τ is the proper time in the case of timelike geodesics, or an affine parameter in the case of null geodesics.

2. By reducing the radial motion of the timelike geodesics to an equation of the form,
$$\tfrac{1}{2}\,\dot{r}^2 + V(r) = \mathcal{E}\,,$$
identify the effective potential $V(r)$. Sketch $V(r)$ and discuss the possible trajectories for massive particles.

3. Consider a freely falling massive particle in the spacetime (1) moving in the radial direction from a point $r = r_* > 0$ with energy E such that $E^2 - \left(\frac{r_*^2}{\ell^2} - \frac{r_0^2}{r_*^2}\right) > 0$. Calculate the proper time that it takes for such a particle to reach $r = 0$.

   Hint: you may use, without proof, that
$$\int dr \frac{1}{\sqrt{a^2 - b^2 r^2 + \frac{c^2}{r^2}}} = -\frac{1}{2b}\arctan\left(\frac{a^2 - 2b^2 r^2}{2b\sqrt{c^2 + a^2 r^2 - b^2 r^4}}\right)\,,$$
   where a, b, c are real and positive constants.

10. Consider the following spacetime

$$ds^2 = -dt^2 + t^{2p_1}\, dx^2 + t^{2p_2}\, dy^2 + t^{2p_3}\, dz^2 \,, \tag{2}$$

where p_i, $i = 1, \ldots, 3$ are real constants. You may assume $t > 0$.

1. By direct computation or otherwise, show that the only non-vanishing Christoffel symbols of the metric (2) are

$$\Gamma^t{}_{xx} = p_1\, t^{-1+2p_1} \,, \quad \Gamma^t{}_{yy} = p_2\, t^{-1+2p_2} \,, \quad \Gamma^t{}_{zz} = p_3\, t^{-1+2p_3} \,,$$
$$\Gamma^x{}_{tx} = \frac{p_1}{t} \,, \quad \Gamma^y{}_{ty} = \frac{p_2}{t} \,, \quad \Gamma^z{}_{tz} = \frac{p_3}{t} \,,$$

   and those related to the above by the symmetries of the Christoffel symbols.

2. Compute the non-vanishing components of the Ricci tensor. (*Hint: by the form of the line element in* (2)*, only the tt, xx, yy and zz components of the Ricci tensor are different from zero. Recall that* $\Gamma^a{}_{ab} = \partial_b \ln \sqrt{|g|}$*, where* $|g| = |\det g_{ab}|$.)

3. Find the conditions that the p_i's must satisfy so that the metric in (2) solves the Einstein vacuum equations, $R_{ab} = 0$.

11. Consider the following spacetime:

$$ds^2 = -\left(1 - \frac{r^2}{\ell^2}\right) dt^2 + \frac{dr^2}{1 - \frac{r^2}{\ell^2}} + r^2 (d\theta^2 + \sin^2\theta\, d\phi^2) \,,$$

where $\ell > 0$ is a constant.

1. Let $u = t - \ell \tanh^{-1}(r/\ell)$ for $r \leq \ell$. Use the coordinates (u, r, θ, ϕ) to show that the surface of $r = \ell$ is non-singular. (*Hint:* Recall that $\frac{d}{dx} \tanh^{-1}(x) = \frac{1}{1-x^2}$.)

2. Show that the vector field $g^{ab}\partial_b u$ is null.

3. Show that the radial null geodesics obey either

$$\frac{du}{dr} = 0 \quad \text{or} \quad \frac{du}{dr} = -\frac{2}{1 - \frac{r^2}{\ell^2}} \,.$$

   For $r < \ell$, which of these families of geodesics is outgoing, i.e., $\frac{dr}{dt} = \frac{\dot{r}}{\dot{t}} > 0$, where the dot $\dot{}$ denotes the derivative with respect to the affine parameter along the geodesics? Sketch the radial null geodesics in the (u, r) plane for $0 \leq r \leq \ell$, where the r-axis is horizontal and the lines of constant u are inclined at $45°$ with respect to the horizontal.