WEEK 9

Previous lecture

- Einstein equations: $G_{ab} = 8\pi G\, T_{ab}$
- In vacuum: $T_{ab} = 0 \Rightarrow R_{ab} = 0$
- Spherically symmetric and static spacetime

$$ds^2 = -e^{2A(r)}dt^2 + e^{2B(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

- Solve the Einstein vacuum eqs
- Step 1: Compute $\Gamma^a{}_{bc}$ → use Euler-Lagrange eqs for the geodesics

$$L = g_{ab}\dot{x}^a\dot{x}^b = -e^{2A(r)}\dot{t}^2 + e^{2B(r)}\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2)$$

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^a}\right) - \frac{\partial L}{\partial x^a} = 0 \iff \ddot{x}^a + \Gamma^a{}_{bc}\dot{x}^b\dot{x}^c = 0$$

· Step 2 : compute the Ricci tensor

$$R_{ab} = \partial_c \Gamma^c{}_{ab} - \partial_a \partial_b \ln\sqrt{|g|} + \Gamma^c{}_{ab} \partial_c \ln\sqrt{|g|} - \Gamma^c{}_{ad} \Gamma^d{}_{bc}$$

$$\det g = - e^{2(A+B)} r^4 \sin^2\theta \rightarrow \sqrt{|g|} = e^{A+B} r^2 \sin\theta$$

$$\rightarrow \ln\sqrt{|g|} = A+B + 2\ln r + \ln \sin\theta$$

· $$R_{tt} = \partial_c \Gamma^c{}_{tt} - \cancelto{0}{\partial_t^2 \ln\sqrt{|g|}} + \Gamma^c{}_{tt} \partial_c \ln\sqrt{|g|} - \Gamma^c{}_{td} \Gamma^d{}_{tc}$$

$$= \partial_r \Gamma^r{}_{tt} + \Gamma^r{}_{tt} \partial_r \ln\sqrt{|g|} - \Gamma^t{}_{tr} \Gamma^r{}_{tt} - \Gamma^r{}_{tt} \Gamma^t{}_{tr}$$

$$= \frac{d}{dr}\left(e^{2(A-B)} A' \right) + e^{2(A-B)} A' \left(A' + B' + \frac{2}{r} \right)$$
$$- 2(A')^2 e^{2(A-B)}$$

$$= e^{2(A-B)} \left[A'' + A'\left(A' - B' + \frac{2}{r}\right) \right]$$

· $$R_{rr} = \partial_c \Gamma^c{}_{rr} - \partial_r^2 \ln\sqrt{|g|} + \Gamma^c{}_{rr} \partial_c \ln\sqrt{|g|} - \Gamma^c{}_{rd} \Gamma^d{}_{rc}$$

$$= \partial_r \Gamma^r{}_{rr} + \partial_r^2 \ln\sqrt{|g|} + \Gamma^r{}_{rr} \partial_r \ln\sqrt{|g|}$$
$$- \Gamma^t{}_{rt} \Gamma^t{}_{rt} - \Gamma^r{}_{rr} \Gamma^r{}_{rr} - \Gamma^\theta{}_{r\theta} \Gamma^\theta{}_{r\theta} - \Gamma^\phi{}_{r\phi} \Gamma^\phi{}_{r\phi}$$

$$= \cancel{B''} - \left(A'' + \cancel{B''} - \cancel{\frac{2}{r^2}} \right) + B'\left(A' + \cancel{B'} + \frac{2}{r} \right)$$
$$- (A')^2 - \cancel{(B')^2} - \cancel{\frac{2}{r^2}}$$

$$= - A'' - (A')^2 + A'B' + \frac{2}{r} B'$$

Similarly we compute

$$R_{\theta\theta} = e^{-2B}\left[r(B'-A')-1\right]+1$$

$$R_{\phi\phi} = \sin^2\theta \, R_{\theta\theta} \quad \text{by spherical symmetry.}$$

· Step 3 : Solve the Einstein equations $R_{ab} = 0$

$$R_{tt} = e^{2(A-B)}\left(A'' + A'^2 - A'B' + \frac{2}{r}A'\right)$$

$$R_{rr} = -A'' - A'^2 + A'B' + \frac{2}{r}B'$$

$$R_{\theta\theta} = e^{-2B}\left[r(B'-A')-1\right]+1$$

Since all components of Ricci have to vanish independently we can consider linear combinations:

$$0 = e^{2(B-A)}R_{tt} + R_{rr} = \frac{2}{r}(A'+B')$$

$$\Rightarrow \quad A(r) = -B(r) + c \quad , \quad c = \text{const}$$

We can set $c=0$ by rescaling $t \to e^{-c}t$ so

$$A(r) = -B(r)$$

$$0 = R_{\theta\theta} = e^{2A(r)}(-2rA'-1)+1$$

$$\Rightarrow \quad e^{2A(r)}(2rA'+1) = 1$$

$$\Rightarrow \quad \frac{d}{dr}\left(r e^{2A(r)}\right) = 1 \quad \Rightarrow \quad e^{2A(r)} = 1 - \frac{R_s}{r}$$

R_s : constant

With these results the line element becomes:

$$ds^2 = -\left(1 - \frac{R_s}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{R_s}{r}} + r^2 d\Omega^2_{(2)}$$

We can fix R_s by requiring that in the weak field regime, $r \gg R_s$, we recover the previous results:

$$g_{tt} = -\left(1 - \frac{R_s}{r}\right) = -(1 + 2\phi) = -\left(1 - \frac{2GM}{r}\right)$$

$$\Rightarrow R_s = 2GM$$

We can finally write down the final form of a static, spherically symmetric vacuum spacetime:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2GM}{r}} + r^2 d\Omega^2_{(2)}$$

→ Schwarzschild metric

M : mass of the spacetime. For $M \to 0$ we recover Minkowski space

- This spacetime describes the exterior of a star. It also describes a black hole

- Birkhoff's Thm : a spherically symmetric solution of the Einstein eqs in vacuum is necessarily static.

· <u>Singularities</u>

The metric coefficients in the Schw. spacetime become 0 or infinity at $r = 0$ and $r = 2GM$, which suggests that something wrong may be going on there. As we shall see $r = 2GM$ is a coordinate singularity, meaning that the coordinates breakdown there. This is similar to $r = 0$ in flat space in polar coordinates: $ds^2 = dr^2 + r^2 d\theta^2$

$\rightarrow$ $g_{\theta\theta} = 0$ and $g^{\theta\theta} = \infty$ @ $r = 0$. But there is nothing wrong at $r = 0$: this point is equivalent to any other point and we can find coordinates, e.g., Cartesian $x = r\cos\theta$, $y = r\sin\theta$ $\rightarrow$ $ds^2 = dx^2 + dy^2$ s.t. the metric and its inverse are perfectly smooth at $x = y = 0$ ($\rightarrow r = 0$).

The singularity at $r = 0$ is of a different nature because curvature scalars, which as such are coordinate invariant, blow up at $r = 0$. This shows that $r = 0$ is a physical singularity since spacetime curvature becomes infinite there. To see this one can compute the so called Kretschmann scalar:

$$R_{abcd}\, R^{abcd} = \frac{48\, G^2 M^2}{r^6}$$

$R_{abcd}\, R^{abcd} \to \infty$ at $r = 0$. On the other hand this invariant is perfectly smooth and finite at $r = 2GM$

Symmetries and Killing vectors.

Suppose that a given metric is independent of a coordinate x^{c*} : $\partial_{c*}\, g_{ab} = 0 \Rightarrow x^{c*} \to x^{c*} + a^{c*}$ is a symmetry. These type of symmetries have a direct consequence for the motion of test particles in g_{ab}, which follow geodesics.

Consider the geodesic equation written in terms of the 4-momentum of the test particle : $p^a = mU^a = m\dot{x}^a$

$\to \quad p^a \nabla_a p^b = 0 \Rightarrow p^a \nabla_a p_b = 0$

$\Rightarrow \quad p^a \partial_a p_b - \Gamma^c{}_{ab}\, p^a p_c = 0$

$$p^a \partial_a p_b = m \frac{dx^a}{d\tau} \partial_a p_b = m \frac{d}{d\tau} p_b$$

$$\Gamma^c{}_{ab} p^a p_c = \frac{1}{2} g^{cd} \left(\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab} \right) p^a p_c$$
$$= \frac{1}{2} \left(\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab} \right) p^a p^d$$
$$= \frac{1}{2} \left(\partial_b g_{ad} \right) p^a p^d$$

$$\Rightarrow \quad m \frac{d}{d\tau} p_b = \frac{1}{2} \left(\partial_b g_{ad} \right) p^a p^d$$

→ If g_{ab} is independent of x^{c*} the p_{c*} is conserved along the geodesic:

$$\partial_{c*} g_{ab} = 0 \quad \Rightarrow \quad \frac{d}{d\tau} p_{c*} = 0 \quad \Rightarrow \quad p_{c*} = \text{const.}$$

Consider the vector $K = \partial_{c*}$ associated to the coordinate x^{c*} that g_{ab} is independent of.

In components, $K^a = (\partial_{c*})^a = \delta^a_{c*}$

We say that K^a generates a symmetry; this means that the transformation under which the geometry is invariant is expressed as a motion in the direction of K^a. We have

$$p_{c*} = K^a p_a = K_a p^a$$

On the other hand,

$$\frac{dP_\alpha}{d\tau} = 0 \qquad \text{by geodesic equation}$$

$$\Rightarrow 0 = p^a \nabla_a (K_b p^b) = p^a \cancel{K_b \nabla_a p^b}^{\,0} + p^a p^b \nabla_a K_b$$

$$= p^a p^b \nabla_{(a} K_{b)}$$

So, if a vector K^a satisfies $\nabla_{(a} K_{b)} = 0$ then $K_b p^b$ is conserved along the geodesics:

$$p^a \nabla_a (K_b p^b) = 0$$

→ A vector K^a that satisfies $\nabla_{(a} K_{b)} = 0$ is called "Killing vector".

→ We have seen how this works for translations but it is more general

· Geodesics of Schwarzschild

The classical experimental tests of GR (i.e., before LIGO) are based on the trajectories of freely falling particles (the planets) and light rays in the gravitational field of a central body (the Sun) → timelike and null geodesics

$$\underset{G=1}{\mathcal{L}} = g_{ab}\,\dot{x}^a\dot{x}^b = -\left(1-\frac{2M}{r}\right)\dot{t}^2 + \frac{\dot{r}^2}{1-\frac{2M}{r}} + r^2\left(\dot{\theta}^2+\sin^2\theta\,\dot{\phi}^2\right)$$

$\mathcal{L}=-1$ for timelike geodesics and $\mathcal{L}=0$ for null geods.

Euler–Lagrange eqs: $\frac{d}{d\lambda}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^a}\right)-\frac{\partial\mathcal{L}}{\partial x^a}=0$

$a=t$: $\ddot{t}+\frac{2M}{r(r-2M)}\dot{t}\dot{r}=0$

$a=r$: $\ddot{r}+\frac{M}{r^3}(r-2M)\dot{t}^2-\frac{M}{r(r-2M)}\dot{r}^2-(r-2M)\left(\dot{\theta}^2+\sin^2\theta\,\dot{\phi}^2\right)=0$

$a=\theta$: $\ddot{\theta}+\frac{2}{r}\dot{\theta}\dot{r}-\sin\theta\cos\theta\,\dot{\phi}^2=0$

$a=\phi$: $\ddot{\phi}+\frac{2}{r}\dot{\phi}\dot{r}+2\cot\theta\,\dot{\theta}\dot{\phi}=0$

- Because the Schw. spacetime is spherically symmetric there is no loss of generality in restricting the motion to the equatorial plane and so we choose

$$\theta=\frac{\pi}{2}$$

- Since g_{ab} is independent of t then $(\partial_t)_a\dot{x}^a$ is constant along the geodesics $\longrightarrow$ conservation of the energy.

$\frac{\partial\mathcal{L}}{\partial t}=0 \Rightarrow \frac{\partial\mathcal{L}}{\partial\dot{t}}=\text{const}$

equivalent to $T^a = (\partial_t)^a \Rightarrow T_a = g_{ab}(\partial_t)^b = -\left(1-\frac{2M}{r}\right)(dt)_a$

$$\Rightarrow \quad T_a \dot{x}^a = -\left(1-\frac{2M}{r}\right)\dot{t} = -E = \text{const}$$

· g_{ab} is independent of $\phi \Rightarrow (\partial_\phi)_a \dot{x}^a = \text{const}$

→ conservation of the angular momentum

Equivalent to : $\frac{\partial L}{\partial \phi} = 0 \Rightarrow \frac{\partial L}{\partial \dot{\phi}} = \text{const}$

$\theta = \pi/2$

$$R^a = (\partial_\phi)^a \rightarrow R_a = g_{ab} R^b = g_{ab}(\partial_\phi)^b = g_{a\phi} = r^2 (d\phi)_a$$

$$\Rightarrow \quad R_a \dot{x}^a = r^2 \dot{\phi} = L = \text{const}$$

Therefore, we have

$$\dot{t} = \frac{E}{1-2M/r}, \qquad \dot{\phi} = \frac{L}{r^2}$$

$$\Rightarrow \quad L = -\frac{E^2}{1-2M/r} + \frac{\dot{r}^2}{1-2M/r} + \frac{L^2}{r^2} = -\varepsilon \qquad (*)$$

with $\varepsilon = 1, 0, -1$ for timelike, null and spacelike geodesics.

Multiply $(*)$ by $\frac{1}{2}\left(1-\frac{2M}{r}\right)$ to get

$$\frac{1}{2}\dot{r}^2 + V(r) = \mathcal{E} \qquad (**)$$

$$V(r) = \frac{1}{2}\left(1-\frac{2M}{r}\right)\left(\frac{L^2}{r^2} + \varepsilon\right) = \frac{1}{2}\varepsilon - \frac{\varepsilon M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3}$$

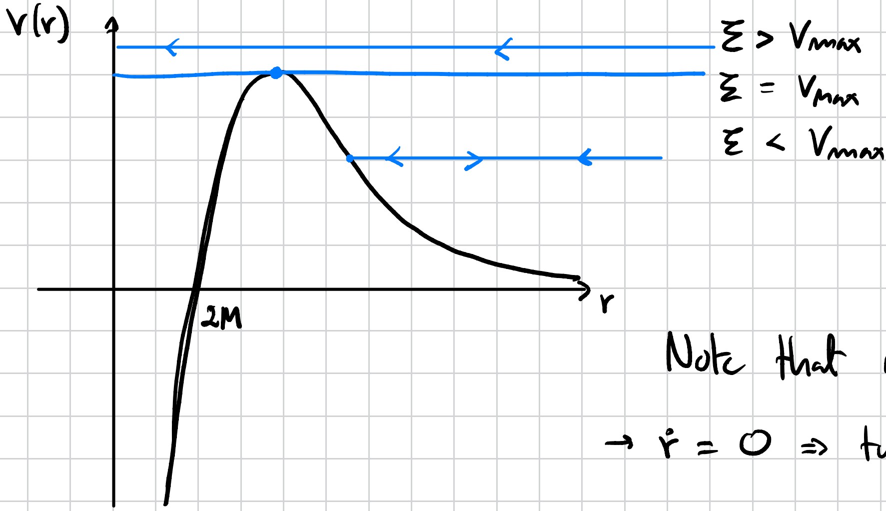
$$\mathcal{E} = \frac{1}{2}E$$

(**) is the equation of motion for a classical particle of unit mass moving in a potential $V(r)$ with energy $\mathcal{E}$.

- We want to get the full trajectory $r(\lambda)$, $t(\lambda)$, $\phi(\lambda)$ but understanding the radial motion provides very good intuition about the possible trajectories.

- $V(r)$ is called " effective potential ".
- In Newtonian gravity one gets a similar equation for the radial motion but with a different $V(r)$: the last term $(\sim 1/r^3)$ is missing, which implies that the motion for small r is different.
- The possible orbits can be obtained by comparing $\mathcal{E}$ to $V(r)$ for different values of L

- Null geodesics: $V(r) = \frac{1}{2}\left(1-\frac{2M}{r}\right)\frac{L^2}{r^2} = \frac{L^2}{2r^2} - \frac{ML^2}{r^3}$



Note that when $\mathcal{E} = V(r)$

$\rightarrow \dot{r} = 0 \Rightarrow$ turning point

To find V_{max},

$$V'(r) = 0 \quad \Rightarrow \quad -L^2 r_c^2 + 3ML^2 = 0 \Rightarrow r_c = 3M$$

Since $\dot{r} = 0 \rightarrow$ the radius is constant so this corresponds to an (unstable) circular orbit.

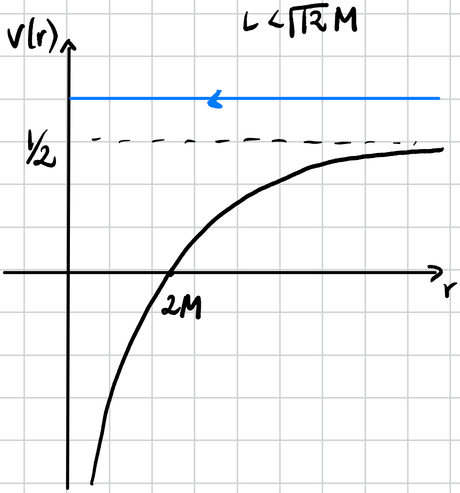
Massive particles :

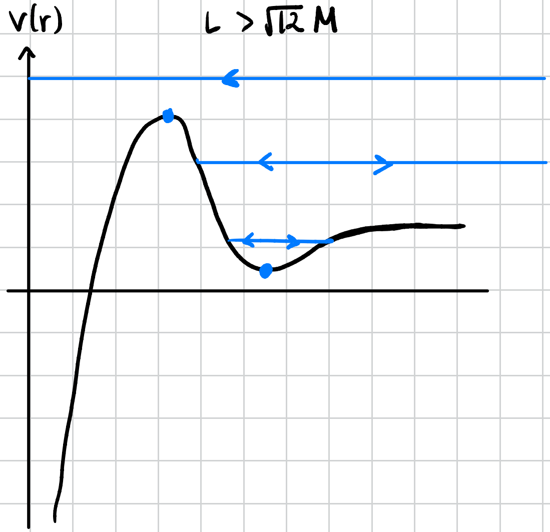
$$V(r) = \frac{1}{2}\left(1 - \frac{2M}{r}\right)\left(\frac{L^2}{r^2} + 1\right) = \frac{1}{2} - \frac{M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3}$$

$\rightarrow$ The shape of $V(r)$ depends on L : extrema only exist for $L > \sqrt{12}\, M$

$$V'(r_c) = 0 \quad \Rightarrow \quad r_c = \frac{L^2 \pm \sqrt{L^4 - 12M^2L^2}}{2M}$$

$\Rightarrow$ For $L > \sqrt{12}\, M$ there exist two circular orbits and for $L < \sqrt{12}\, M$ there are no circular orbits





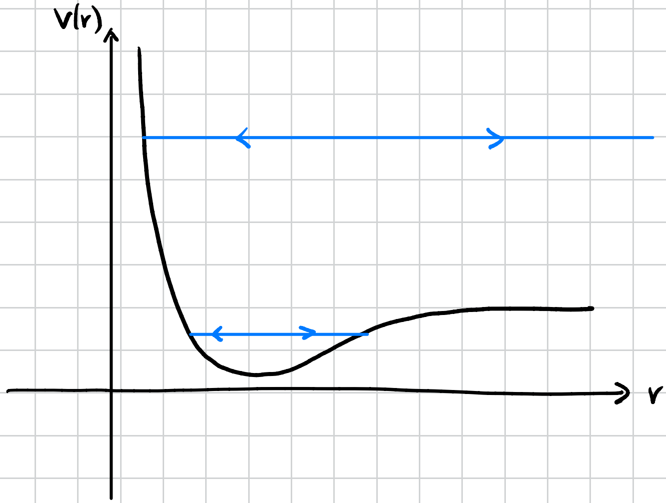
- Fall to $r = 0$ for all E

- Fall to $r = 0$
- Circular orbits at r_c: unstable and stable
- Hyperbolic orbits
- Closed orbits (ellipses)

For $L = \sqrt{12}\, M$ the two circular orbits come together at $r_c = 6M$ and they disappear for $L < \sqrt{12}\, M$. So $r_c = 6M$ is the smallest possible radius for a stable circular orbit for a massive particle in Schw. So in Schw. there are stable circular orbits for $r > 6M$ and unstable circular orbits for $3M < r < 6M$. This is for free particles. Accelerating particles can dip below $r = 3M$ and escape to infinity as long as they don't get below $r = 2M$.

- Contrast with Newtonian potential

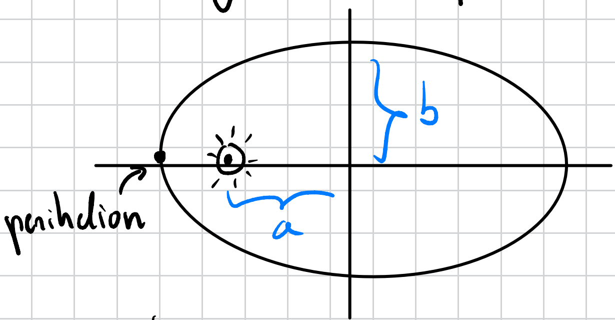
$$V(r) = \frac{1}{2} - \frac{M}{r} + \frac{L^2}{2r^2}$$



→ It's impossible to reach $r = 0$!

Experimental tests of GR

- **Perihelion precession**
- Perihelion: point of closest approach to the centre of an ellipse



→ Want to calculate $r(\phi)$ for closed orbits of massive particles

Consider the radial motion of a massive particle in Schw:

$$\frac{1}{2}\dot{r}^2 + V(r) = \mathcal{E} \qquad (*)$$

$$V(r) = \frac{1}{2}\left(1-\frac{2M}{r}\right)\left(\frac{L^2}{r^2}+1\right) = \frac{1}{2} - \frac{M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3}$$

$$\mathcal{E} = \frac{1}{2}E$$

Also, recall that $r^2\dot{\phi} = L \Rightarrow \frac{1}{\dot{\phi}^2} = \frac{r^4}{L^2}$

To get an equation for $\frac{dr}{d\phi}$, consider

$$\frac{dr}{d\phi} = \frac{d\tau}{d\phi}\frac{dr}{d\tau} = \frac{\dot r}{\dot\phi} \Rightarrow \left(\frac{dr}{d\phi}\right)^2 = \frac{\dot r^2}{\dot\phi^2}$$

Multiplying $(*)$ by $\frac{1}{\dot\phi^2} = \frac{r^4}{L^2}$

$$\left(\frac{dr}{d\phi}\right)^2 + \frac{r^4}{L^2} - \frac{2M}{L^2}r^3 + r^2 - 2Mr = \frac{2\mathcal{E}}{L^2}r^4$$

Define the new dimensionless radial variable

$$u = \frac{L^2}{Mr} \rightarrow \frac{du}{d\phi} = -\frac{L^2}{Mr^2}\frac{dr}{d\phi} = -\frac{u^2 M}{L^2}\frac{dr}{d\phi}$$

$$\rightarrow \frac{dr}{d\phi} = -\frac{L^2}{Mu^2}\frac{du}{d\phi}$$

$$\frac{L^4}{M^2u^4}\left(\frac{du}{d\phi}\right)^2 + \frac{L^6}{M^4u^4} - \frac{2L^4}{M^2u^3} + \frac{L^4}{M^2u^2} - \frac{2L^2}{u} = \frac{2\mathcal{E}L^6}{M^4u^2}$$

$$\left(\frac{du}{d\phi}\right)^2 + \frac{L^2}{M^2} - 2u + u^2 - \frac{2M^2}{L^2}u^3 = \frac{2\mathcal{E}L^2}{M^2}$$

Differentiating again with respect to ϕ:

$$2\left(\frac{du}{d\phi}\right)\frac{d^2u}{d\phi^2} - 2\frac{du}{d\phi} + 2u\frac{du}{d\phi} - \frac{6M^2}{L^2}u^2\frac{du}{d\phi} = 0$$

$$\Rightarrow \frac{d^2u}{d\phi^2} - 1 + u = \underbrace{\frac{3M^2}{L^2}u^2}_{\text{GR term}}$$

· We will solve this equation treating the GR term perturbatively:

$$u = u_0 + \varepsilon u_1 , \qquad \varepsilon \ll 1 \qquad \frac{3M^2}{L^2} \sim \mathcal{O}(\varepsilon)$$

Taylor-expanding and collecting the various orders in ε we find:

$\mathcal{O}(\varepsilon^0)$:
$$\frac{d^2}{d\phi^2} u_0 - 1 + u_0 = 0$$

$$\Rightarrow \quad u_0 = 1 + e\cos\phi \quad \Rightarrow \quad r = \frac{L^2}{M(1+e\cos\phi)}$$

→ That's the equation for an ellipse with eccentricity e.

$\mathcal{O}(\varepsilon)$:

$$\frac{d^2}{d\phi^2} u_1 + u_1 = \frac{3M^2}{L^2} u_0^2 = \frac{3M^2}{L^2} (1 + e\cos\phi)$$
$$= \frac{3M^2}{L^2}\left[\left(1 + \frac{1}{2}e^2\right) + 2e\cos\phi + \frac{1}{2}e^2\cos(2\phi)\right]$$

$$\Rightarrow \quad u_1 = \frac{3M^2}{L^2}\left[\left(1+\frac{1}{2}e^2\right) + e\,\phi\sin\phi - \frac{1}{6}e^2\cos(2\phi)\right]$$

The first term is a shift in u and the last term oscillates around 0. The second term is a secular term and alters the period of the orbits.

To see this, combine this term with the 0^{th} order solution:

$$u = 1 + e\cos\phi + \frac{3M^2}{L^2} e\,\phi\sin\phi$$
$$= 1 + e\cos[(1-\alpha)\phi] \quad , \quad \alpha = \frac{3M^2}{L^2}$$

Indeed, Taylor-expand in α:

$$\cos[(1-\alpha)\phi] \approx \cos\phi + \alpha \frac{d}{d\alpha}\cos[(1-\alpha)\phi]\Big|_{\alpha=0} + O(\alpha^2)$$
$$= \cos\phi + \alpha\,\phi\sin\phi + O(\alpha^2)$$

→ This implies that the periodicity of u is no longer 2π but

$$(1-\alpha)\Delta\phi = 2\pi \Rightarrow \Delta\phi = \frac{2\pi}{1-\alpha} \approx 2\pi(1+\alpha)$$

⇒ for each orbit the perihelion advances

$$\delta\phi = 2\pi\alpha = \frac{6\pi G^2 M^2}{L^2}$$

L can be related to the parameters of the 0^{th} order ellipse: The equation for an ellipse with semi-major axis $\underline{a}$ is

$$r = \frac{(1-e^2)a}{1+e\cos\phi} = \frac{L^2}{M(1+e\cos\phi)} \Rightarrow L^2 = Ma(1-e^2)$$

$$\Rightarrow \quad \delta\phi = \frac{6\pi GM}{c^2(1-e^2)a}$$

For Mercury this is $\delta\phi = 43''/\text{century}$ and for the Earth $\delta\phi = 3.8''/\text{century}$.

· <u>Bending of light</u>

· Repeating the exact same computation as before but now for null geodesics $(\varepsilon = 0)$ and defining $u = 1/r$ we get the following equation for u:

$$\frac{d^2u}{d\phi^2} + u = \underbrace{\frac{3GM}{c^2}}_{\text{small GR correction}} u^2$$

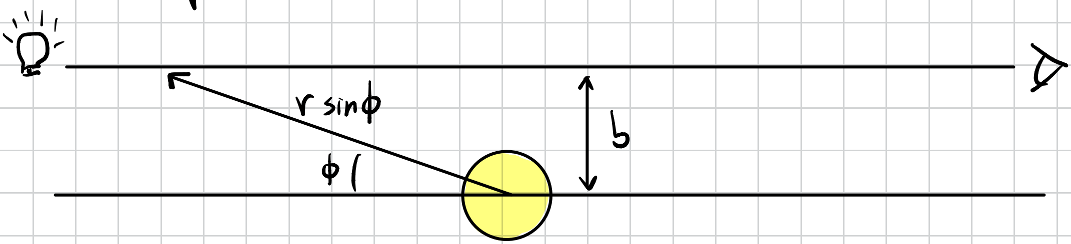
$$\delta = \frac{GM}{c^2} \ll u$$

As before, we expand the solution in perturbation theory: $u = u_0 + u_1$, $u_1 \sim \delta$

0^{th} order: $\frac{d^2}{d\phi^2} u_0 + u_0 = 0$

$\Rightarrow \quad u_0 = \frac{1}{b} \sin\phi \quad \Rightarrow \quad r \sin\phi = b$

→ Straight line with impact parameter b : light sent from $r = \infty$ ($\phi = 0$) returns to $r = \infty$ ($\phi = \pi$)



→ Change of the angle ϕ along the trajectory: $\Delta\phi = \pi$

The equation for the perturbation is

$$\frac{d^2 u_1}{d\phi^2} + u_1 = 3\delta u_0^2 = \frac{3\delta}{b^2} \sin^2\phi$$

$\Rightarrow \quad u_1 = \frac{\delta}{b^2} \left(1 + C \cos\phi + \cos^2\phi\right)$; C : integration constant

So the full solution including the first order corrections is

$$u = \frac{1}{b} \sin\phi + \frac{\delta}{b^2} \left(1 + C \cos\phi + \cos^2\phi\right)$$

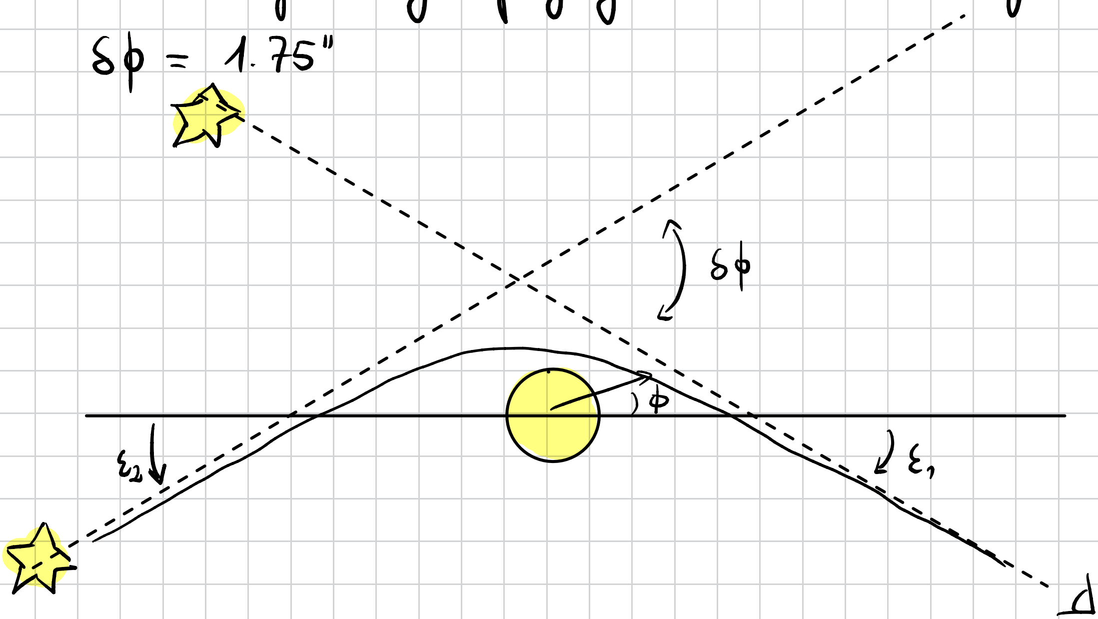
We want to calculate the deflection angle $\delta\phi$.

Far away from the source, $r \to \infty$ and hence $u \to 0$ and we take the angle ϕ to be $-\varepsilon_1$ and $\pi + \varepsilon_2$ for $r \to \infty$, with ε_1 and ε_2 $O(\delta)$

$$-\frac{\varepsilon_1}{b} + \frac{\delta}{b^2}(2+C) = 0 \quad ; \quad -\frac{\varepsilon_2}{b} + \frac{\delta}{b^2}(2-C) = 0$$

$$\Rightarrow \quad \delta\phi = \varepsilon_1 + \varepsilon_2 = \frac{4\delta}{b} = \frac{4GM}{c^2 b}$$

For a light ray grazing the Sun this gives
$\delta\phi = 1.75''$



- Gravitational Redshift

Consider a stationary observer in the Schw. geometry: the four-velocity is U^a with $U^i = 0$

$$\Rightarrow \quad U_a U^a = g_{ab} U^a U^b = -1$$

$$\Rightarrow \quad U^0 = \left(1 - \frac{2M}{r}\right)^{-1/2}$$

The frequency of a photon ($\rightarrow$ null geodesic $x^a(\lambda)$) measured by such an observer is

$$\omega = -g_{ab} U^a \dot{x}^b \quad , \quad \dot{x}^a = \frac{dx^a}{d\lambda} : \text{ tangent vector to the geodesic}$$

$$= \left(1 - \frac{2M}{r}\right)^{1/2} \dot{t}$$

$$= \left(1 - \frac{2M}{r}\right)^{-1/2} E \quad \text{ since } \left(1 - \frac{2M}{r}\right)\dot{t} = E = \text{const}$$

along the geodesics

Since E is constant, then ω will take different values when measured at different values of r: For a photon emitted at r_1 and observed at r_2, the corresponding measured frequencies are

$$\frac{\omega_2}{\omega_1} = \left(\frac{1 - 2M/r_1}{1 - 2M/r_2}\right)^{1/2}$$

In the limit $r_1, r_2 \gg 2M$, we have

$$\frac{\omega_2}{\omega_1} \approx 1 - \frac{M}{r_1} + \frac{M}{r_2} = 1 + \Phi_1 - \Phi_2$$

where $\Phi = -\frac{M}{r}$ $(G=1)$ is the Newtonian potential.