MTH5114 Linear Programming and Game Theory, Spring 2024

Week 4 Coursework Questions **Viresh Patel**

These exercises should be completed individually and submitted (together with those of weeks 5 and 6) via the course QMPlus page by **9am on Monday, 11 March**.

Make sure you clearly write your **name** and **student ID** number at the top of your submission. Note that the credit available for each week's coursework exercises is roughly equal.

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1. Suppose we have a linear program in standard equation form

$$\begin{aligned} \text{maximize} \quad & \mathbf{c}^\mathsf{T}\mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

and suppose $\mathbf{u}$, $\mathbf{v}$. and $\mathbf{w}$ are all optimal solutions to this linear program.

(a) Prove that $\frac{1}{3}\mathbf{u} + \frac{1}{3}\mathbf{v} + \frac{1}{3}\mathbf{w}$ is a feasible solution.

(b) Prove that $\frac{1}{3}\mathbf{u} + \frac{1}{3}\mathbf{v} + \frac{1}{3}\mathbf{w}$ is an optimal solution.

(c) Your proofs for (b) and (c) should work more generally for certain linear combinations of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. State for which linear combinations of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ your proofs still work. (You do not have to justify your answer for part (c)).

Hint: for this question it is helpful to look at certain parts of the proofs covered in week 4 and to adapt them.

Solution:

(a) Note that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are feasible solutions (since they are optimal solutions). Let $\mathbf{z} = \frac{1}{3}\mathbf{u} + \frac{1}{3}\mathbf{v} + \frac{1}{3}\mathbf{w}$. We must show that $\mathbf{z}$ is feasible, i.e. that $A\mathbf{z} = \mathbf{b}$ and that $\mathbf{z} \geq \mathbf{0}$. The first of these holds because

$$A\mathbf{z} = A(\tfrac{1}{3}\mathbf{u} + \tfrac{1}{3}\mathbf{v} + \tfrac{1}{3}\mathbf{w}) = \tfrac{1}{3}A\mathbf{u} + \tfrac{1}{3}A\mathbf{v} + \tfrac{1}{3}A\mathbf{w} = \tfrac{1}{3}\mathbf{b} + \tfrac{1}{3}\mathbf{b} + \tfrac{1}{3}\mathbf{b} = \mathbf{b},$$

where we used that $A\mathbf{u} = A\mathbf{v} = A\mathbf{w} = \mathbf{b}$ since $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are feasible.

To show that $\mathbf{z} \geq 0$, looking at the ith coordinate, we have

$$\mathbf{z}_i = \tfrac{1}{3}\mathbf{u}_i + \tfrac{1}{3}\mathbf{v}_i + \tfrac{1}{3}\mathbf{w}_i \geq 0$$

where we used that $\mathbf{u}_i \geq 0$, $\mathbf{v}_i \geq 0$, and $\mathbf{w}_i \geq 0$ since $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are feasible. Since this holds for all coordinates, then $\mathbf{z} \geq \mathbf{0}$.

(b) We already know $\mathbf{z}$ is feasible. The easiest way to show $\mathbf{z}$ is optimal is to show e.g. that $\mathbf{c}^\mathsf{T}\mathbf{z} = \mathbf{c}^\mathsf{T}\mathbf{u}$ (since we know $\mathbf{u}$ is optimal)[1]. Since $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are all optimal, we know they must have the same objective value, i.e. $\mathbf{c}^\mathsf{T}\mathbf{u} = \mathbf{c}^\mathsf{T}\mathbf{v} = \mathbf{c}^\mathsf{T}\mathbf{w}$.

Now we have

$$\mathbf{c}^\mathsf{T}\mathbf{z} = \mathbf{c}^\mathsf{T}(\tfrac{1}{3}\mathbf{u}+\tfrac{1}{3}\mathbf{v}+\tfrac{1}{3}\mathbf{w}) = \tfrac{1}{3}\mathbf{c}^\mathsf{T}\mathbf{u}+\tfrac{1}{3}\mathbf{c}^\mathsf{T}\mathbf{v}+\tfrac{1}{3}\mathbf{c}^\mathsf{T}\mathbf{w} = \tfrac{1}{3}\mathbf{c}^\mathsf{T}\mathbf{u}+\tfrac{1}{3}\mathbf{c}^\mathsf{T}\mathbf{u}+\tfrac{1}{3}\mathbf{c}^\mathsf{T}\mathbf{u} = \mathbf{c}^\mathsf{T}\mathbf{u}$$

where the third equality uses $\mathbf{c}^\mathsf{T}\mathbf{u} = \mathbf{c}^\mathsf{T}\mathbf{v} = \mathbf{c}^\mathsf{T}\mathbf{w}$.

(c) The proofs above work for any vector $\lambda\mathbf{u} + \mu\mathbf{v} + \nu\mathbf{w}$ where $\lambda, \mu, \nu \geq 0$ and $\lambda + \mu + \nu = 1$.

[1] It would also be fine e.g. to show that $\mathbf{c}^\mathsf{T}\mathbf{z} = \mathbf{c}^\mathsf{T}\mathbf{v}$ or $\mathbf{c}^\mathsf{T}\mathbf{z} = \mathbf{c}^\mathsf{T}\mathbf{w}$

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Week 5 Coursework Questions **Viresh Patel**

These exercises should be completed individually and submitted (together with those of weeks 4 and 6) via the course QMPlus page by **9am on Monday, 11 March**.

Make sure you clearly write your **name** and **student ID** number at the top of your submission:.

Solve the following linear program using the simplex algorithm. You should give the initial tableau and each further tableau produced during the execution of the algorithm. If the program has an optimal solution, give this solution and state its objective value. If it does not have an optimal solution, say why.

You should indicate the highlighted row and columns in each pivot step as well as the row operations you carry out. This is in order to gain credit even if the final answer is incorrect.

1.
$$\begin{aligned} \text{maximize} \quad & 2x_1 + 3x_2 + 5x_3 + x_4 \\ \text{subject to} \quad & x_1 + x_2 + 2x_3 + x_4 \leq 2, \\ & 3x_2 + 3x_3 + 3x_4 \leq 6, \\ & 3x_1 + 2x_2 + 2x_3 + x_4 \leq 7, \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

Solution:

The following tableaux are produced.

	x_1	x_2	x_3	x_4	s_1	s_2	s_3	
s_1	1	1	2	1	1	0	0	2
s_2	0	3	3	3	0	1	0	6
s_3	3	2	2	1	0	0	1	7
$-z$	2	3	5	1	0	0	0	0

	x_1	x_2	x_3	x_4	s_1	s_2	s_3	
x_3	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	1
s_2	$-\frac{3}{2}$	$\frac{3}{2}$	0	$\frac{3}{2}$	$-\frac{3}{2}$	1	0	3
s_3	2	1	0	0	-1	0	1	5
$-z$	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{3}{2}$	$-\frac{5}{2}$	0	0	-5

	x_1	x_2	x_3	x_4	s_1	s_2	s_3	
x_1	1	1	2	1	1	0	0	2
s_2	-3	0	-3	0	-3	1	0	0
s_3	1	0	-2	-1	-2	0	1	3
$-z$	-1	0	-1	-2	-3	0	0	-6

Here, the algorithm terminates. The optimal solution is $x_1 = 0, x_2 = 2, x_3 = 0, s_1 = 0, s_2 = 0, s_3 = 3$. This solution has objective value 6.

2. Suppose that we are carrying out the simplex algorithm on a linear program in standard inequality form (with 3 variables and 4 constraints) and suppose that we have reached a point where we have obtained the following tableau. Apply one more pivot operation, indicating the highlighted row and column and the row operations you carry out. What can you conclude from your updated tableau?

	x_1	x_2	x_3	s_1	s_2	s_3	s_4	
s_1	-2	0	1	1	0	0	0	3
s_2	3	0	-2	0	1	2	0	6
x_2	1	1	-3	0	0	1	0	2
s_4	-3	0	2	0	0	-1	1	4
$-z$	-2	0	11	0	0	-4	0	-8

Solution: We highlight the third column and the fourth row and obtain the following tableau. If the four rows of the given tableau are R_1, R_2, R_3, R_4, then we replace these rows respectively with $R_1 - \frac{1}{2}R_4$, $R_2 + R_4$, $R_3 + \frac{3}{2}R_4$, $\frac{1}{2}R_4$.

	x_1	x_2	x_3	s_1	s_2	s_3	s_4	
s_1	$-\frac{1}{2}$	0	0	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	1
s_2	0	0	0	0	1	1	1	10
x_2	$-\frac{7}{2}$	1	0	0	0	$-\frac{1}{2}$	$\frac{3}{2}$	8
x_3	$-\frac{3}{2}$	0	1	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	2
$-z$	$\frac{29}{2}$	0	0	0	0	$\frac{3}{2}$	$-\frac{11}{2}$	-30

At this point (with the updated table above) we pick x_1 as the leaving variable (i.e. we highlight the first column) and find that there is no bound when we try to choose a leaving variable (i.e. there is no positive entry in the highlighted column). Thus, the linear program is unbounded and has no optimal solution.

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Week 6 Coursework Questions **Viresh Patel**

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It is generally recommended (here and in the exam) that in your answers, as well as giving the tableaux, you also highlight the appropriate rows and columns and mention the row operations you are carrying out – this way, even if your final answer is incorrect, we can easily award you credit for using the right method.

1. Solve the following linear programmne using the 2-phase simplex algorithm. You should give the initial tableau and each further tableau produced during the execution of the algorithm. If the program has an optimal solution, give this solution and state its objective value. If it does not have an optimal solution, say why.

$$\begin{aligned} \text{maximize} \quad & x_1 - 2x_2 + x_3 - 4x_4 \\ \text{subject to} \quad & 2x_1 + x_2 - 2x_3 - x_4 \geq 1, \\ & 5x_1 + x_2 - x_3 - x_4 \leq -1, \\ & 2x_1 + x_2 - x_3 - 3x_4 \geq 2, \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

Solution: The linear program after transforming into standard equation form and then adding artificial variables is

$$\begin{aligned} \text{maximize} \quad & x_1 - 2x_2 + x_3 - 4x_4 \\ \text{subject to} \quad & -2x_1 - x_2 + 2x_3 + x_4 + s_1 - a_1 = -1, \\ & 5x_1 + x_2 - x_3 - x_4 + s_2 - a_2 = -1, \\ & -2x_1 - x_2 + x_3 + 3x_4 + s_3 - a_3 = -2, \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

The initial tableau is

	x_1	x_2	x_3	x_4	s_1	s_2	s_3	a_1	a_2	a_3	
a_1	-2	-1	2	1	1	0	0	-1	0	0	-1
a_2	5	1	-1	-1	0	1	0	0	-1	0	-1
a_3	-2	-1	1	3	0	0	1	0	0	-1	-2
$-w$	0	0	0	0	0	0	0	-1	-1	-1	0
$-z$	1	-2	1	-4	0	0	0	0	0	0	0

Bringing this tableau into valid form, we obtain the following tableau.

	x_1	x_2	x_3	x_4	s_1	s_2	s_3	a_1	a_2	a_3	
a_1	2	1	-2	-1	-1	0	0	1	0	0	1
a_2	-5	-1	1	1	0	-1	0	0	1	0	1
a_3	2	1	-1	-3	0	0	-1	0	0	1	2
$-w$	-1	1	-2	-3	-1	-1	-1	0	0	0	4
$-z$	1	-2	1	-4	0	0	0	0	0	0	0

Continuing with phase 1, we obtain the following tableau.

	x_1	x_2	x_3	x_4	s_1	s_2	s_3	a_1	a_2	a_3	
x_2	2	1	-2	-1	-1	0	0	1	0	0	1
a_2	-3	0	-1	0	-1	-1	0	1	1	0	2
a_3	0	0	1	-2	1	0	-1	-1	0	1	1
$-w$	-3	0	0	-2	0	-1	-1	-1	0	0	3
$-z$	5	0	-3	-6	-2	0	0	2	0	0	2

Here, the first phase ends. We see that $-w \neq 0$, so the given program is infeasible and has no optimal solution.

2. Apply the first phase of the 2-phase simplex algorithm to the following linear programme giving the initial tableau and each further tableau produced. Give the starting tableau for the second phase if there is one.

$$\begin{aligned} \text{maximize} \quad & 2x_1 + x_2 + 3x_3 \\ \text{subject to} \quad & x_2 - x_3 \leq 2, \\ & x_1 + 3x_2 + 2x_3 \geq 3, \\ & 2x_1 + 2x_2 + x_3 = 4, \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

Solution: The linear programme after transforming to standard equation form and introducing artificial variables becomes

$$\begin{aligned}\text{maximize}\quad & 2x_1 + x_2 + 3x_3\\ \text{subject to}\quad & x_2 - x_3 + s_1 = 2,\\ & -x_1 - 3x_2 - 2x_3 + s_2 - a_1 = -3,\\ & 2x_1 + 2x_2 + x_3 + a_2 = 4,\\ & x_1, x_2, x_3, s_1, s_2, a_1, a_2 \geq 0.\end{aligned}$$

The initial tableau is

	x_1	x_2	x_3	s_1	s_2	a_1	a_2	
s_1	0	1	-1	1	0	0	0	2
a_1	-1	-3	-2	0	1	-1	0	-3
a_2	2	2	1	0	0	0	1	4
$-w$	0	0	0	0	0	-1	-1	0
$-z$	2	1	3	0	0	0	0	0

Bringing the initial tableau into valid form, we obtain:

	x_1	x_2	x_3	s_1	s_2	a_1	a_2	
s_1	0	1	-1	1	0	0	0	2
a_1	1	3	2	0	-1	1	0	3
a_2	2	2	1	0	0	0	1	4
$-w$	3	5	3	0	-1	0	0	7
$-z$	2	1	3	0	0	0	0	0

The following tableaux are produced for the first phase:

	x_1	x_2	x_3	s_1	s_2	a_1	a_2	
s_1	$-\frac{1}{3}$	0	$-\frac{5}{3}$	1	$\frac{1}{3}$	$-\frac{1}{3}$	0	1
x_2	$\frac{1}{3}$	1	$\frac{2}{3}$	0	$-\frac{1}{3}$	$\frac{1}{3}$	0	1
a_2	$\frac{4}{3}$	0	$-\frac{1}{3}$	0	$\frac{2}{3}$	$-\frac{2}{3}$	1	2
$-w$	$\frac{4}{3}$	0	$-\frac{1}{3}$	0	$\frac{2}{3}$	$-\frac{5}{3}$	0	2
$-z$	$\frac{5}{3}$	0	$\frac{7}{3}$	0	$\frac{1}{3}$	$-\frac{1}{3}$	0	-1

	x_1	x_2	x_3	s_1	s_2	a_1	a_2	
s_1	0	0	$-\frac{7}{4}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{2}$
x_2	0	1	$\frac{3}{4}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{2}$
x_1	1	0	$-\frac{1}{4}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{2}$
$-w$	0	0	0	0	0	-1	-1	0
$-z$	0	0	$\frac{11}{4}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{5}{4}$	$-\frac{7}{2}$

Here, we finish the first phase. We see that the linear programme is feasible, since $-w = 0$. The tableaux for the second phase are as given below. (Note that for the coursework, you are only required to give the first tableau, but the rest are given for completeness.)

	x_1	x_2	x_3	s_1	s_2	
s_1	0	0	$-\frac{7}{4}$	1	$\frac{1}{2}$	$\frac{3}{2}$
x_2	0	1	$\frac{3}{4}$	0	$-\frac{1}{2}$	$\frac{1}{2}$
x_1	1	0	$-\frac{1}{4}$	0	$\frac{1}{2}$	$\frac{3}{2}$
$-z$	0	0	$\frac{11}{4}$	0	$-\frac{1}{2}$	$-\frac{7}{2}$

	x_1	x_2	x_3	s_1	s_2	
s_1	0	$\frac{7}{3}$	0	1	$-\frac{2}{3}$	$\frac{8}{3}$
x_3	0	$\frac{4}{3}$	1	0	$-\frac{2}{3}$	$\frac{2}{3}$
x_1	1	$\frac{1}{3}$	0	0	$\frac{1}{3}$	$\frac{5}{3}$
$-z$	0	$-\frac{11}{3}$	0	0	$\frac{4}{3}$	$-\frac{16}{3}$

	x_1	x_2	x_3	s_1	s_2	
s_1	2	3	0	1	0	6
x_3	2	2	1	0	0	4
s_2	3	1	0	0	1	5
$-z$	-4	-5	0	0	0	-12

Here the algorithm terminates. The optimal solution is $x_3 = 4, s_1 = 6, s_2 = 5$ and $x_1, x_2 = 0$. This solution has objective value 12.