MTH5113 (2023/24): Problem Sheet 8

All coursework should be submitted individually.

- Problems marked "[**Marked**]" should be submitted and will be marked.

Please submit the completed problem on QMPlus:

- At the portal for **Coursework Submission 2**.

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(1) *(Warm-up)* For each of the following parts:

(i) Sketch the surface S.

(ii) Draw the unit normal $\mathbf{n_p}$ on the sketch from part (i).

(iii) Give an informal description (e.g. "outward-facing", "inward-facing", "upward-facing") of the side of S represented by the normal $\mathbf{n_p}$.

(a) $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$, and

$$\mathbf{n_p} = (0, 1, 0)_{(0,-1,0)}.$$

(b) $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$, and

$$\mathbf{n_p} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)_{\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{3}{4}\right)}.$$

(c) $S = \{(x, y, z) \in \mathbb{R}^3 \mid z = x^2 + y^2\}$, and

$$\mathbf{n_p} = \left(\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3}\right)_{(1,-1,2)}.$$

(2) *(Warm-up)* Compute the surface areas of the following parametric surfaces:

(a) *Parametric torus:*

$$\alpha : (0, 2\pi) \times (0, 2\pi) \to \mathbb{R}^3, \qquad \alpha(u, v) = ((2 + \cos u) \cos v,\ (2 + \cos u) \sin v,\ \sin u).$$

(See Question (8b) of Problem Sheet 1 for a plot of α.)

(b) *Parallellogram spanned by vectors* $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$*:*

$$\beta : (0,1) \times (0,1) \to \mathbb{R}^3, \qquad \beta(u,v) = u \cdot \mathbf{a} + v \cdot \mathbf{b}.$$

State your answer in terms of $\mathbf{a}$ and $\mathbf{b}$.

(3) [**Marked**] Consider the surface of revolution

$$\mathcal{H} = \left\{ (x,y,z) \in \mathbb{R}^3 \mid x^2 + z^2 = \left(\frac{3}{2} + \cos \frac{y}{2} \right)^2 \right\}.$$

(a) Find the tangent plane to $\mathcal{H}$ at $\left(\frac{3}{2\sqrt{2}}, \pi, -\frac{3}{2\sqrt{2}} \right)$. *(See also (Q7) of Problem Sheet 7.)*

(b) Find the unit normals to $\mathcal{H}$ at $\left(\frac{3}{2\sqrt{2}}, \pi, -\frac{3}{2\sqrt{2}} \right)$.

(c) Which of the two unit normals in (b) represents the "outward-facing" side of $\mathcal{H}$?

(For part (c), you do not have to prove the answer. You can find the answer by sketching $\mathcal{H}$ *and the appropriate normals and then inspecting your sketch.)*

(4) [**Tutorial**] Consider the sphere

$$\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

(a) Find two parametrisations of $\mathbb{S}^2$ such that the combined images of all these parametrisations cover all of $\mathbb{S}^2$.

(b) Show that the unit normals to $\mathbb{S}^2$ at any $\mathbf{p} \in \mathbb{S}^2$ are given by $\pm \mathbf{p}_\mathbf{p}$.

(c) What choice of unit normals of $\mathbb{S}^2$ defines the "outward-facing" orientation of $\mathbb{S}^2$? What choice of unit normals of $\mathbb{S}^2$ defines the "inward-facing" orientation of $\mathbb{S}^2$?

(5) *(Fun with graphs)* Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function, and let

$$G_f = \{(x,y,z) \in \mathbb{R}^3 \mid z = f(x,y)\}$$

be the graph of f, which we know to be a surface. For any $(x,y) \in \mathbb{R}^2$:

(a) Find the tangent plane to G_f at $(x, y, f(x,y))$.

(b) Find the unit normals to G_f at $(x, y, f(x, y))$.

Give your answers in terms of f *and its derivatives at* (x, y).

(6) *(Tangent planes revisited)* Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function, and let

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = 0\}$$

be a level set of f. In addition, assume $\nabla f(\mathbf{p})$ is nonzero for any $\mathbf{p} \in S$, so that S is a surface. Show that at each $\mathbf{p} \in S$, the tangent plane to S at $\mathbf{p}$ satisfies

$$T_{\mathbf{p}}S = \{\mathbf{v}_{\mathbf{p}} \in T_{\mathbf{p}}\mathbb{R}^3 \mid \mathbf{v}_{\mathbf{p}} \cdot \nabla f(\mathbf{p}) = 0\}.$$

(7) *(Surface area in higher dimensions)*

(a) Let $\mathcal{P}$ be a parallelogram in $\mathbb{R}^n$, with two of its sides given by tangent vectors $\mathbf{a}_{\mathbf{p}}$ and $\mathbf{b}_{\mathbf{p}}$ (where $\mathbf{a}, \mathbf{b}, \mathbf{p} \in \mathbb{R}^n$). Recall from lectures and the lecture notes that when $n = 3$, the area of $\mathcal{P}$ is given by $|\mathbf{a} \times \mathbf{b}|$. Show that for general n, the area of $\mathcal{P}$ satisfies

$$\mathcal{A}(\mathcal{P}) = \sqrt{\det \begin{bmatrix} \mathbf{a} \cdot \mathbf{a} & \mathbf{a} \cdot \mathbf{b} \\ \mathbf{b} \cdot \mathbf{a} & \mathbf{b} \cdot \mathbf{b} \end{bmatrix}}.$$

(In particular, when $n \neq 3$*, we no longer have the cross product.)*

(b) Use the results from part (a) to give a reasonable definition of the surface area of a regular parametric surface $\sigma : U \to \mathbb{R}^n$, for any dimension n.

(8) *(Confusion with Möbius bands)* Consider the parametric surface

$$\sigma : (-1, 1) \times \mathbb{R} \to \mathbb{R}^3, \qquad \sigma(u, v) = \left(\left(1 - \frac{u}{2} \sin \frac{v}{2}\right) \cos v,\ \left(1 - \frac{u}{2} \sin \frac{v}{2}\right) \sin v,\ \frac{u}{2} \cos \frac{v}{2} \right),$$

and let M be defined as the image of σ. One can, in fact, show that M is a surface, and that σ is a parametrisation of M whose image is all of M. *(Here, you can assume both of these facts without proving them.)* In particular, this M gives an explicit description of a *Möbius band*; see Figure 4.21 in the lecture notes for an illustration of M.

Ms. Mistake (who is close friends with Mr. Error from Problem Sheet 4) decides to choose the following unit normals to M:

$$\mathbf{n}^+_\sigma(u,v) = +\left[\frac{\partial_1\sigma(u,v)\times\partial_2\sigma(u,v)}{|\partial_1\sigma(u,v)\times\partial_2\sigma(u,v)|}\right]_{\sigma(u,v)}, \qquad (u,v)\in(-1,1)\times\mathbb{R}.$$

Ms. Mistake concludes that the $\mathbf{n}^+_\sigma(u,v)$'s she chose define an orientation of M, and hence M is orientable! As a wise tutor for *MTH5113*, explain why Ms. Mistake is mistaken!