§10 Power series

§10.1 Power series

Given a sequence $\{a_n\}_{n=0}^{\infty}$ of real numbers, the series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is called a <u>power series</u>. in the variable $x \in \mathbb{R}$.

Every power series converges for $x = 0$, $f(0) = a_0$. The convergence for other values of x depends on the the nature of the sequence $\{a_n\}_{n=0}^{\infty}$.

Def^n 10.1.1 (Radius of convergence)

The radius of convergence R of the power series $\sum_{n=0}^{\infty} a_n x^n$ is given by

$$R = \sup\{|x| \mid \sum_{n=0}^{\infty} a_n x^n \text{ converges}\} \qquad R \geq 0$$

Note $R \geq 0$ as the series always converges for $x=0$. If the series converges for all $x \in \mathbb{R}$, we often write "$R=\infty$" – infinite radius of convergence.

REMINDER!

Series: Ratio Test & (Cauchy) Root Test: $\sum_{n=0}^{\infty} a_n$ (Conv / Divg.)

Ratio Test $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

Root Test $\beta = \limsup_{n\in\mathbb{N}} |a_n|^{1/n}$

$$\begin{cases} <1 \to \sum_{n=1}^{\infty} a_n \ (C) \\ >1 \to \sum_{n=1}^{\infty} a_n \ (D) \end{cases}$$

- Recall Ratio Test for Series $\sum a_n$

  If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists, then it is $\beta = \lim_{n\to\infty} |a_n|^{1/n}$

- The Ratio Test is usually easier to use than the Root Test.

- Investigation of convergence for $|x| = R$ has to be treated separately for both $x = R$ and $x = -R$.

Move from $\sum a_n$ to $\sum a_n x^n$. – n-th term $a_n x^n$.

Theorem 10.1.2 Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series and define: $\beta = \limsup_{n\in\mathbb{N}} |a_n|^{1/n}$, and $R = 1/\beta$.

(If $\beta = 0$, we say $R = \infty$ and if $\beta = \infty$, $R = 0$).

Then we have:

the power series converg for all $|x| < R$,

" diverges for all $|x| > R$.

Note - no conclusion about $|x| = R$, $x = R$, $x = -R$

Proof We apply the "root test" to the series $\sum a_n x^n$.

For each $x \in \mathbb{R}$, consider

$$\alpha_x = \limsup_{n\to\infty} |a_n x^n|^{1/n}$$

$$\limsup_{n\to\infty} |a_n|^{1/n} |x| = \beta |x|$$

$\left(\beta = \limsup_{n\in\mathbb{N}} a_n^{1/n}\right)$

[For comparison, $\lim_{n\to\infty} \left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| |x| = \gamma |x|$

where $\gamma = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.]

(i) Suppose $0 < R < \infty$, then $\alpha_x = |x|\beta = \frac{|x|}{R}$

If $|x| < R$, then $\alpha_x < 1$ and series converges by Root Test

Similarly, if $|x| > R \Leftrightarrow \alpha_x > 1$ and series diverges

(ii) Suppose $R = \infty$, then $\beta = 0$ & $\alpha_x = 0 < 1$, and series converges for all $x \in \mathbb{R}$ e.g. $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

(iii) Suppose that $R = 0$, $\beta = +\infty$, $\alpha_x = +\infty$, power series diverges for all $x \neq 0$. e.g. $\sum_{n=0}^{\infty} n!\, x^n$

Comment

$$f(x) = T_{(n,0)}f(x) + R_{(n,0)}f(x)$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\sum_{k=0}^{n} f^{(k)}(0)\frac{x^k}{k!} + f^{(n+1)}(c)\frac{x^{n+1}}{(n+1)!}$$

Note the necessary "balance" in this equation when $n \to \infty$ is considered

$$f(x) = \lim_{n\to\infty} \sum_{k=0}^{n} f^{(k)}(0)\frac{x^k}{k!} = \sum_{k=0}^{\infty} f^{(k)}(0)\frac{x^k}{k!} \iff \lim_{n\to\infty} R_{(n,0)}f(x) = 0$$

Example 10.1.3 Consider the series $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$

$a_n = \frac{1}{n!}$, and using the ratio test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0 \quad \therefore \quad R = +\infty$$

$\therefore$ power series converges for all $x \in \mathbb{R}$.

in fact $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

Example 10.1.4 Consider the series $\sum_{n=0}^{\infty} x^n$

$a_n = 1, \forall n \Rightarrow \beta = 1, R = 1. \Rightarrow \sum_{n=0}^{\infty} x^n$ converges for $|x| < 1$

Note for $|x| = R = 1$, $x = +/- 1$, $\sum (1)^n$ & $\sum (-1)^n$ both diverge

$\sum_{n=0}^{\infty} (1)$ - partial sums diverge to ∞.

$\sum_{n=0}^{\infty} (-1)^n$ - partial sums alternate between 1 and 0.

Harmonic series diverges

(i) For $x=1$: $\sum_1 \frac{1}{n} = 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3}+\frac{1}{4}\right)}_{\geq \frac{1}{2}} + \underbrace{\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)}_{\geq \frac{1}{2}}$

$+ \underbrace{\left(\frac{1}{9}+\cdots+\frac{1}{16}\right)}_{\geq \frac{1}{2}} + \underbrace{\left(\frac{1}{17}+\cdots \frac{1}{32}\right)}_{\geq \frac{1}{2}} + \cdots$

We see $\sum_{n=1}^{2^k} \frac{1}{n} > 1 + k\left(\frac{1}{2}\right)$ which tends $\to \infty$ as $k \to \infty$.

$\therefore \lim_{k\to\infty} \sum_{n=1}^{2^k} \frac{1}{n} = \infty$

(ii) For $x=-1$

$\sum_1 \frac{(-1)^n}{n}$ is an alternating series and $\left|\frac{(-1)^n}{n}\right| \to 0$

as $n \to \infty \Rightarrow$ convergence

Example 10.1.7 $\sum_{n=0}^{\infty} n!\,x^n$, $B=\infty$, $R=0$

Converges at $x=0$, diverges $x\neq 0$.

These examples show that all 4 types of interval of convergence can occur:

e.g. $(-R,R)$, $[-R,R]$, $[-R,R)$, $(-R,R]$

If we consider power series of the form

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n$$

with x_0 fixed, then we can get (correspondingly) all 4 types centred on $x=x_0$, i.e.

(x_0-R, x_0+R), $[x_0-R, x_0+R]$, etc.