Reminder

Coursework 2 deadline: Monday 18 March
9 am

Recap quiz

How do we know when to stop the basic simplex algorithm? (week 5)

We stop the basic simplex algorithm if, in our current tableau, all the entries in the last row are zero or negative

How do we know when phase 1 ends in the 2-phase simplex algorithm?

We stop phase 1 if, in our current tableau, all the entries in the w-row are zero or negative

If the far right entry in w-row is 0 we move on to phase 2 and otherwise we conclude our LP is infeasible.

So far
- modelling real-life problem
- solving LPs using geometry
- extreme points / basic feasible solutions
- Simplex = solving LPs algebraically

Duality

Every LP has a "twin" LP called the dual
The dual has several practical interpretations.
Later will see that solving these two LPs together models two people playing a game.

Motivating example

$$
\begin{aligned}
\text{maximise} \quad & 2x_1 + 3x_2 + x_3 \\
\text{subject to} \quad & x_1 + x_2 + x_3 \leq 10 \\
& \tfrac{1}{2}x_1 + x_2 \leq 8 \\
& x_1 + x_2 - x_3 \leq 4 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

Want a quick upper bound for the objective using constraints.

Motivating example

$$\begin{array}{llr}
\text{maximise} & 2x_1 + 3x_2 + x_3 & \\
\text{subject to} & x_1 + x_2 + x_3 \le 10 & C_1 \\
& \tfrac{1}{2}x_1 + x_2 \le 8 & C_2 \\
& x_1 + x_2 - x_3 \le 4 & C_3 \\
& x_1, x_2, x_3 \ge 0 &
\end{array}$$

Want a quick upper bound for the objective using constraints.

Consider $3C_1$. Gives

$$\underbrace{2x_1 + 3x_2 + x_3}_{\text{objective}} \underset{\uparrow\, x_1, x_2, x_3 \ge 0}{\le} 3x_1 + 3x_2 + 3x_3 \underset{\uparrow\, 3C_1}{\le} 30$$

So every feasible solution has objective value ≤ 30.

Give better bound by taking linear combination of constraints.

Take $C_1 + 2C_2$. Gives

$$(x_1 + x_2 + x_3) + 2\left(\tfrac{1}{2}x_1 + x_2\right) \le 10 + 2 \times 8 = 26$$

$$\Rightarrow \quad \underbrace{2x_1 + 3x_2 + x_3}_{= \text{ objective.}} \le 26$$

Every feasible solution has objective value ≤ 26.

Here, not allowed to consider e.g. $4C_1 - C_3$. Why?

The minus sign reverses inequality in C_3.

What is the best possible upper band we can get?

$$\begin{array}{llr}
\text{maximise} & 2x_1 + 3x_2 + x_3 & \\
\text{subject to} & x_1 + x_2 + x_3 \leq 10 & C_1 \\
 & \frac{1}{2}x_1 + x_2 \leq 8 & C_2 \\
 & x_1 + x_2 - x_3 \leq 4 & C_3 \\
 & x_1, x_2, x_3 \geq 0 &
\end{array}$$

Consider $y_1 C_1 + y_2 C_2 + y_3 C_3$ with $y_1, y_2, y_3 \geq 0$

Gives $y_1(x_1 + x_2 + x_3) + y_2(\frac{1}{2}x_1 + x_2) + y_3(x_1 + x_2 - x_3)$

$$\underset{(*)}{\leq} 10y_1 + 8y_2 + 4y_3$$

i.e $(y_1 + \frac{1}{2}y_2 + y_3)x_1 + (y_1 + y_2 + y_3)x_2 + (y_1 - y_3)x_3$

$$\underset{(*)}{\leq} 10y_1 + 8y_2 + 4y_3$$

Then we have

$$2x_1 + 3x_2 + x_3 \underset{(**)}{\overset{\text{want}}{\leq}} (y_1 + \frac{1}{2}y_2 + y_3)x_1 + (y_1 + y_2 + y_3)x_2 + (y_1 - y_3)x_3$$

$$\leq 10y_1 + 8y_2 + 4y_3$$

Best possible upper band on objective given by solving

$$\begin{array}{llr}
\text{minimise} & 10y_1 + 8y_2 + 4y_3 & \\
\text{subject to} & y_1 + \frac{1}{2}y_2 + y_3 \geq 2 & C_1' \\
 & y_1 + y_2 + y_3 \geq 3 & C_2' \\
 & y_1 - y_3 \geq 1 & C_3' \\
 & y_1, y_2, y_3 \geq 0 &
\end{array}$$

constraints guarantee (**)

guarantees (*)

How would we modify

(i) if x_1 was unrestricted C' has to be equality constraint

(ii) C_3 was equality constraint. y_3 is unrestricted.

Defn Given LP in standard inequality form with n variables and m constraints

$$\begin{aligned} \max\ & \underline{c}^T \underline{x} \\ \text{sub to}\ & A\underline{x} \le \underline{b} \\ & \underline{x} \ge \underline{0} \end{aligned}$$

i.e.

$$\begin{aligned} \max\ & c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{sub to}\ & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2 \\ & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m \\ & x_1, x_2, \ldots, x_n \ge 0 \end{aligned}$$

the dual LP has m variables and n constraints and is given by

$$\begin{aligned} \min\ & \underline{b}^T \underline{y} \\ \text{sub to}\ & A^T\underline{y} \ge \underline{c} \\ & \underline{y} \ge \underline{0} \end{aligned}$$

$$\begin{aligned} \min\ & b_1y_1 + b_2y_2 + \cdots + b_my_m \\ \text{sub to}\ & a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \ge c_1 \\ & a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m \ge c_2 \\ & \vdots \\ & a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \ge c_n \\ & y_1, y_2, \ldots, y_m \ge 0. \end{aligned}$$

Example

What is dual of

$$\begin{aligned} \max\ & 10x_1 + 20x_2 \\ \text{sub to}\ & x_1 + 2x_2 \le 3 \\ & 4x_1 + 5x_2 \le 9 \\ & x_1, x_2 \ge 0 \end{aligned}$$

dual

$$\begin{aligned} \min\ & 3y_1 + 9y_2 \\ \text{sub to}\ & y_1 + 4y_2 \ge 10 \\ & 2y_1 + 5y_2 \ge 20 \\ & y_1, y_2 \ge 0 \end{aligned}$$

Note: original LP is called the primal LP

<u>Thm</u> (Weak duality theorem for LPs)

Consider a LP in standard inequality form

i.e. max $\underline{c}^T\underline{x}$ and its dual min $\underline{b}^T\underline{y}$

sub to $A\underline{x} \le \underline{b}$ sub to $A^T\underline{y} \ge \underline{c}$

$\underline{x} \ge \underline{0}$ $\underline{y} \ge \underline{0}$

If $\underline{x}$ is a feasible solution for the primal LP and $\underline{y}$ is a feasible solution for the dual LP

then $\underline{c}^T\underline{x} \le \underline{b}^T\underline{y}$

<u>Example</u>

What is dual of

max $10x_1 + 20x_2$

sub to $x_1 + 2x_2 \le 3$

$4x_1 + 5x_2 \le 9$

$x_1, x_2 \ge 0$

dual

min $3y_1 + 9y_2$

sub to $y_1 + 4y_2 \ge 10$

$2y_1 + 5y_2 \ge 20$

$y_1, y_2 \ge 0$

Find any feasible solution for primal and any feasible solution for dual and verify that the weak duality theorem holds.

e.g. for primal $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is feasible

objective value (i.e. $\underline{c}^T\underline{x}$) $= 10\times1 + 20\times1 = 30$

e.g. for dual $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$ is feasible

objective value i.e. $\underline{b}^T\underline{y} = 3\times0 + 9\times5 = 45$

Indeed $\underline{c}^T\underline{x} \le \underline{b}^T\underline{y}$.

<u>Thm</u> (Weak duality theorem for LPs)

Consider a LP in standard inequality form

i.e. $\max \underline{c}^T\underline{x}$ sub to $A\underline{x} \leq \underline{b}$, $\underline{x} \geq \underline{0}$ and its dual $\min \underline{b}^T\underline{y}$ sub to $A^T\underline{y} \geq \underline{c}$, $\underline{y} \geq \underline{0}$

If $\underline{x}$ is a feasible solution to the primal LP and $\underline{y}$ is a feasible solution for the dual LP

then $\underline{c}^T\underline{x} \leq \underline{b}^T\underline{y}$

<u>Pf</u> <u>Fact 1</u> Suppose $\underline{a}, \underline{b}, \underline{x} \in \mathbb{R}^n$ with $\underline{a} \leq \underline{b}$, $\underline{x} \geq \underline{0}$. Then

(a) $\underline{a}^T \leq \underline{b}^T$ (b) $\underline{x}^T\underline{a} \leq \underline{x}^T\underline{b}$

<u>Fact 2</u> For matrices $(AB)^T = B^T A^T$

Know $\underline{A\underline{x} \leq \underline{b}}^{(1)}$ and $\underline{x} \geq \underline{0}$ since $\underline{x}$ is feasible for primal

$A^T\underline{y} \geq \underline{c}$ and $\underline{y} \geq \underline{0}$ since $\underline{y}$ is feasible for dual

Same as $(A^T\underline{y})^T \geq \underline{c}^T$ by Fact 1(a)

same as $\underline{\underline{y}^T A \geq \underline{c}^T}^{(2)}$ by Fact 2

Multiply (1) by $\underline{y}^T$ on the left to give

$\underline{y}^T A\underline{x} \leq \underline{y}^T\underline{b}$ by Fact 1(b) since $\underline{y} \geq \underline{0}$

Multiply (2) by $\underline{x}$ on the right to give

$\underline{y}^T A\underline{x} \geq \underline{c}^T\underline{x}$ by Fact 1(b) since $\underline{x} \geq \underline{0}$

Combining gives $\underline{c}^T\underline{x} \leq \underline{y}^T A\underline{x} \leq \underline{y}^T\underline{b} = \underline{b}^T\underline{y}$ $\square$

Thm (strong duality theorem)

Suppose we have LP in standard inequality form

$$\max\ \underline{c}^T \underline{x} \qquad \text{and its dual} \qquad \min\ \underline{b}^T \underline{y}$$

$$\text{sub to } A\underline{x} \leq \underline{b} \qquad\qquad \text{sub to } \underline{A}^T \underline{y} \geq \underline{c}$$

$$\underline{x} \geq \underline{0} \qquad\qquad \underline{y} \geq \underline{0}$$

If $\underline{x}^*$ is an optimal solution for the primal LP

$\underline{y}^*$ is an optimal solution for the dual LP

$$\text{then} \quad \underline{c}^T \underline{x}^* = \underline{b}^T \underline{y}^*$$

i.e. both LPs have the same optimal objective value.

Proof

Goal: find some feasible $\underline{y}$ for the dual

s.t. $\underline{b}^T \underline{y} = \underline{c}^T \underline{x}^*$

If we can do this then

$$\underline{b}^T \underline{y}^* \geq \underline{c}^T \underline{x}^* = \underline{b}^T \underline{y} \geq \underline{b}^T \underline{y}^*$$

weak duality theorem.

because y^* is optimal

$\Rightarrow$ all inequalities above are equalities

$\Rightarrow$ $\underline{b}^T \underline{y}^* = \underline{c}^T \underline{x}^*$ as required.

<u>Goal</u>: find some feasible $\underline{y}$ for the dual

s.t. $\underline{b}^T \underline{y} = \underline{c}^T \underline{x}^*$

We will achieve our goal by applying Simplex to primal LP.

Consider initial and final tableaux

Initial tableau

		x_1	x_2	x_i	x_n	s_1	s_2	$\cdots$	s_m	
R_1	s_1			a_{1i}		1	0		0	b_1
R_2	s_2		A	a_{2i}		0	1		$\vdots$	b_2
	$\vdots$					0	0	$\ddots$	0	$\vdots$
R_m	s_m			a_{mi}		0	0		1	b_m
R_z		c_1	c_2	c_i	c_n	0	0	$\cdots$	0	0

Final tableau

		x_1	x_2	x_i	x_n	s_1	s_2	$\cdots$	s_m	
R_1'										
R_2'	?		?				?			?
$\vdots$										
R_m'										
R_z'		p_1	p_2	p_i	p_n	q_1	q_2	$\cdots$	q_m	z^*

Overall, we know R_z' is obtained from R_z by adding a linear combination of the other rows

We can read it off

$$R_z' = R_z + q_1 R_1 + q_2 R_2 + \cdots + q_m R_m \quad (*)$$

Choose $\underline{y} \in \mathbb{R}^m$ s.t. $y_i = -q_i$

<u>Claim</u> $\underline{y}$ is feasible for the dual

<u>Pf of claim</u> $\underline{y} \geq \underline{0}$ since $q_i \leq 0 \ \forall i$ since all entries in last row of final tableau are ≤ 0.

Examine column of x_i

using $(*)$ we have that

$$c_i + a_{1i}\underset{-y_1}{q_1} + a_{2i}\underset{-y_2}{q_2} + \cdots + a_{mi}\underset{-y_m}{q_m} = p_i \leq 0$$

final row of final tableau.

$$\Rightarrow c_i \leq a_{1i} y_1 + a_{2i} y_2 + \cdots + a_{mi} y_m$$

This shows $\underline{y}$ satisfies i^{th} constraint of dual. Works for all $i = 1, \ldots, n$

So $\underline{y}$ is feasible //

<u>Claim</u> $\underline{b}^T \underline{y} = \underline{c}^T \underline{x}^*$

<u>Pf of claim</u> Note that $\underline{c}^T \underline{x}^* = -z^*$

Examining final column using $(*)$

$$q_1 b_1 + q_2 b_2 + \cdots q_m b_m + 0 = z^*$$

$$\Rightarrow -y_1 b_1 - y_2 b_2 - \cdots - y_m b_m = z^* = -\underline{c}^T \underline{x}^*$$

$$\Rightarrow -\underline{b}^T \underline{y} = -\underline{c}^T \underline{x}^*$$

$$\Rightarrow \underline{b}^T \underline{y} = \underline{c}^T \underline{x}^*.$$

// $\square$

Remarks

1) In proof, we assumed $\underline{b} \geq \underline{0}$ so that we could apply standard simplex algorithm. Essentially the same argument works if we apply the 2-phase algorithm.

2) When applying simplex to primal LP the last row also gives optimal solution for dual (take $y_i = -q_i$).

If LP is not in standard inequality form, how do we find the dual?

Assume LP is maximisation (something similar for minimisation (see printed notes).

1) Multiply constraint by -1 if necessary so that each constraint looks like

$$a_1x_1 + a_2x_2 + \dots + a_nx_n \leq b$$

or $$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$$

Replace each variable x_i with $-\bar{x}_i$ if necessary so all sign restrictions are ≥ 0 or unrestricted.

2) Recall dual of standard inequality form:

Primal

max $c_1x_1 + c_2x_2 + \dots + c_nx_n$

i.e. subtc $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1$ con 1

$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \leq b_2$ con 2

$\vdots$

$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_n$ con m

$x_1, x_2, \dots, x_n \geq 0$

dual

min $b_1y_1 + b_2y_2 + \dots + b_my_m$

subtc $a_{11}y_1 + a_{21}y_2 + \dots + a_{m1}y_m \geq c_1$ con 1'

$a_{12}y_1 + a_{22}y_2 + \dots + a_{m2}y_m \geq c_2$ con 2'

$\vdots$

$a_{1n}y_1 + a_{2n}y_2 + \dots + a_{mn}y_m \geq c_n$ con n'

$y_1, y_2, \dots, y_m \geq 0$

After step 1 our LP looks like primal LP above except some constraints have equality and some variables are unrestricted.

If any constraint in primal LP, say con i, is an equality constraint then replace $y_i \geq 0$ with y_i unrestricted in dual.

If any variable in primal, say x_j, is unrestricted then con j' with equality in dual

Example Find dual of following LP

$$\begin{array}{llllll}
\text{maximise} & x_1 & + 6x_2 & + 5x_3 & & \\
\text{sub to} & x_1 & & - 3x_3 & & \leq 10 \\
& & x_2 & & + 7x_4 & \geq 13 \\
& 7x_1 & + 11x_2 & + 9x_3 & & = 100 \\
& 3x_1 & - 5x_2 & + 7x_3 & & \leq 20
\end{array}$$

$x_1, x_4 \geq 0$ x_2, x_3 unrestricted.

Step 1

$$\begin{array}{lllllll}
\text{maximise} & x_1 & + 6x_2 & + 5x_3 & & & \\
\text{sub to} & x_1 & & - 3x_3 & & \leq 10 & y_1 \\
& & - x_2 & & - 7x_4 & \leq -13 & y_2 \\
& 7x_1 & + 11x_2 & + 9x_3 & & = 100 & y_3 \\
& 3x_1 & - 5x_2 & + 7x_3 & & \leq 20 & y_4
\end{array}$$

$x_1, x_4 \geq 0$ x_2, x_3 unrestricted,

Step 2

$$\begin{array}{lllllll}
\text{minimise} & 10y_1 & - 13y_2 & + 100y_3 & + 20y_4 & & \\
\text{sub to} & y_1 & & + 7y_3 & + 3y_4 & \geq 1 & x_1 \geq 0 \\
& 0y_1 & - y_2 & + 11y_3 & - 5y_4 & = 6 & x_2 \text{ unres} \\
& -3y_1 & & + 9y_3 & + 7y_4 & = 5 & x_3 \text{ unres} \\
& & -7y_2 & & & \geq 0 & x_4 \geq 0
\end{array}$$

$y_1, y_2, y_4 \geq 0$, y_3 unres.