Mathematical Tools for Asset Management

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Problem Set: Factor Models and APT

7. Consider the following two factor model for returns of three stocks:

$$r_A = 0.13 + 6F_1 + 4F_2 + \varepsilon_A$$
$$r_B = 0.15 + 2F_1 + 2F_2 + \varepsilon_B$$
$$r_C = 0.07 + 5F_1 - F_2 + \varepsilon_C$$

Assume that factors and epsilons have means of zero. Also, assume the factors have variances of 0.01 and uncorrelated with each other. If $\mathrm{var}(\varepsilon_A) = 0.01$, $\mathrm{var}(\varepsilon_B) = 0.04$ and $\mathrm{var}(\varepsilon_C) = 0.02$, what are the variances of the returns of the three stocks as well as the covariances and correlations between them? What are the expected returns of the three stocks?

$$\sigma_A^2 = 6^2\ \mathrm{var}(F_1) + 4^2\ \mathrm{var}(F_2) + \mathrm{var}(\varepsilon_A)$$
$$= 36(0.01) + 16(0.01) + 0.01$$
$$= 0.53$$

$$\sigma_B^2 = 2^2\ \mathrm{var}(F_1) + 2^2\ \mathrm{var}(F_2) + \mathrm{var}(\varepsilon_B)$$
$$= 4(0.01) + 4(0.01) + 0.04$$
$$= 0.12$$

$$\sigma_C^2 = 5^2\ \mathrm{var}(F_1) + (-1)^2\ \mathrm{var}(F_2) + \mathrm{var}(\varepsilon_c)$$
$$= 25(0.01) + 1(0.01) + 0.02$$
$$= 0.28$$

$$\sigma_{AB} = \operatorname{cov}\left(6F_1 + 4F_2 + \varepsilon_{\mathrm{A}}, 2F_1 + 2F_2 + \varepsilon_{\mathrm{B}}\right)$$
$$= \operatorname{cov}\left(6F_1, 2F_1\right) + \operatorname{cov}\left(4F_2, 2F_2\right)$$
$$= 12\operatorname{var}\left(F_1\right) + 8\operatorname{var}\left(F_2\right)$$
$$= 12\left(0.01\right) + 8\left(0.01\right)$$
$$= 0.20$$

$$\sigma_{BC} = \operatorname{cov}\left(2F_1, 5F_1\right) + \operatorname{cov}\left(2F_2, -1F_2\right)$$
$$= 10\operatorname{var}\left(F_1\right) - 2\operatorname{var}\left(F_2\right)$$
$$= 10\left(0.01\right) - 2\left(0.01\right)$$
$$= 0.08$$

$$\sigma_{AC} = \operatorname{cov}\left(6F_1, 5F_1\right) + \operatorname{cov}\left(4F_2, -1F_2\right)$$
$$= 30\left(0.01\right) - 4\left(0.01\right)$$
$$= 0.26$$

$$\rho_{\mathrm{AB}} = \frac{\sigma_{\mathrm{AB}}}{\sigma_{\mathrm{A}}\sigma_{\mathrm{B}}} = \frac{0.20}{\sqrt{(0.53)(0.12)}} = 0.793$$

$$\rho_{BC} = \frac{\sigma_{BC}}{\sigma_B\sigma_C} = \frac{0.08}{\sqrt{(0.28)(0.12)}} = 0.436$$

$$\rho_{AC} = \frac{\sigma_{AC}}{\sigma_A\sigma_C} = \frac{0.26}{\sqrt{(0.53)(0.28)}} = 0.675$$

$$r_A = 0.13 + 6F_1 + 4F_2 + \varepsilon_A$$
$$E(r_A) = 0.13 + 6(0) + 4(0) + 0 = 0.13 = 13.0\%$$
$$r_B = 0.15 + 2F_1 + 2F_2 + \varepsilon_B$$
$$E(r_B) = 0.15 + 2(0) + 2(0) + 0 = 0.15 = 15.0\%$$
$$r_C = 0.07 + 5F_1 - 1F_2 + \varepsilon_C$$
$$E(r_C) = 0.07 + 5(0) - 1(0) + 0 = 0.07 = 7.0\%$$

8. Write out the factor equations (return of portfolio written as a function of factors) and expected returns of the following portfolios:

a. A portfolio of the three stocks in exercise 1 with $20,000 invested in stock A, $20,000 invested in stock B, and $10,000 invested in stock C.

b. A portfolio consisting of the portfolio formed in part 1 of this exercise and a $3,000 short position in stock C of exercise 1.

9. How much should be invested in each of the stocks in exercise 1 to design the pure factors portfolios or the factor tracking portfolios (Hint: factor 1 tracking portfolio has factor1 beta=1 and factor 2 beta=0.)

Compute the expected returns of these portfolios. Then compute the risk premiums of these two portfolios.

If there exist an additional asset with the following factor equation: $r_4 = 0.08 + F_1 + \varepsilon_4$ does an arbitrage opportunity exists? If so, describe how you would take advantage of it a.

Let x_A be the weight of stock A in the factor 1 tracking portfolio
Let x_B be the weight of stock B in the factor 1 tracking portfolio
Let x_C be the weight of stock C in the factor 1 tracking portfolio

$$6x_A + 2x_B + 5(1 - x_A - x_B) = 1$$
$$4x_A + 2x_B - 1(1 - x_A - x_B) = 0$$

$$x_A - 3x_B = -4$$
$$5x_A + 3x_B = 1$$

$$6x_A = -3$$
$$\text{Portfolio 1: } x_A = -\frac{1}{2}, x_B = \frac{7}{6}, x_C = \frac{1}{3}$$

$$E(R_{p1}) = -\frac{1}{2}(0.13) + \frac{7}{6}(0.15) + \frac{1}{3}(0.07) = 0.1333 = 13.33\%$$

b.

Let x_A be the weight of stock A in the factor 2 tracking portfolio
Let x_B be the weight of stock B in the factor 2 tracking portfolio
Let x_C be the weight of stock C in the factor 2 tracking portfolio

$$6x_A + 2x_B + 5(1 - x_A - x_B) = 0$$
$$4x_A + 2x_B - 1(1 - x_A - x_B) = 1$$

$$x_A - 3x_B = -5$$
$$5x_A + 3x_B = 2$$

$$6x_A = -3$$
$$\text{Portfolio 2}: \ x_A = -\frac{1}{2}, x_B = \frac{3}{2}, x_C = 0$$

$$E(R_{p2}) = -\frac{1}{2}(0.13) + \frac{3}{2}(0.15) = 0.16 = 16.0\%$$

c.

Let x_A be the weight of stock A in the risk free tracking portfolio
Let x_B be the weight of stock B in the risk free tracking portfolio
Let x_C be the weight of stock C in the risk free tracking portfolio

$$6x_A + 2x_B + 5 - 5x_A - 5x_B = 0$$
$$4x_A + 2x_B - 1 + x_A + x_B = 0$$

$$x_A - 3x_B = -5$$
$$5x_A + 3x_B = 1$$

$$6x_A = -4$$

$$x_A = -\frac{2}{3}$$
$$x_B = \frac{13}{9}$$
$$x_C = \frac{2}{9}$$

Risk free rate:

$$R_f = -\frac{2}{3}(0.13) + \frac{13}{9}(0.15) + \frac{2}{9}(0.07) = 0.1456 = 14.56\%$$

Then the risk premiums are:

$$\lambda_1 = 0.1333 - 0.1456 = -0.0123 = -1.23\%$$
$$\lambda_2 = .16 - .1456 = .0144 = 1.44\%$$