Midterm test

Question 1

(a) $(x')^0 = \gamma\left(x^0 - \frac{v}{c} x^1\right)$ with $v = |\vec{v}|$ and

$(x')^1 = \gamma\left(x^1 - \frac{v}{c} x^0\right)$ $\quad x^0 = ct$, $(x')^0 = ct'$

$(x')^2 = x^2$ and $(x')^3 = x^3$

(b) $\frac{\partial x'^a}{\partial x^b} = L^a{}_b$. By using $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \cosh\alpha$

$$L^a{}_b = \begin{pmatrix} \gamma & -\gamma\frac{v}{c} & 0 & 0 \\ -\gamma\frac{v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh\alpha & -\sinh\alpha & 0 & 0 \\ -\sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We have $(x')^a = L^a{}_b x^b$ while dual vectors transform with the inverse matrix

$$(L^{-1})^a{}_b = \begin{pmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$x'_a = x_b \left(L^{-1}\right)^b{}_a$$

(c) $E = \gamma m c^2$ with $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \simeq 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \dots$

$$\Rightarrow \quad E \simeq \underset{(1)}{m c^2} + \underset{(2)}{\frac{1}{2} m v^2} + \underset{(3)}{\frac{3}{8} m \frac{v^4}{c^2}} + \dots$$

The first term is the rest mass energy,
the second term is the Newtonian kinetic energy,
the third term is the first relativistic correction.

We have $\frac{3}{8} m \frac{v^4}{c^2} \cdot \left(\frac{1}{2} m v^2\right) = \frac{3}{4}\frac{v^2}{c^2}$

(2) a) $\mathcal{L}_X(\omega_a Y^a) = X^b \partial_b(\omega_a Y^b)$

$$= X^b(\cancel{\omega_a \partial_b Y^a} + Y^a \partial_b \omega_a) \qquad (1)$$

$$= Y^a \mathcal{L}_X \omega_a + \omega_a \mathcal{L}_X Y^a$$

$$= Y^a \mathcal{L}_X \omega_a + \omega_a (\cancel{X^b \partial_b Y^a} - Y^b \partial_b X^a) \qquad (2)$$

$$\Rightarrow \quad Y^a \mathcal{L}_X \omega_a = X^b Y^a \partial_b \omega_a + \omega_a Y^b \partial_b X^a$$

$$= Y^a (X^b \partial_b \omega_a + \omega_b \partial_a X^b)$$

$$\Rightarrow \quad \mathcal{L}_X \omega_a = X^b \partial_b \omega_a + \omega_b \partial_a X^b \qquad \square$$

b) $\mathcal{L}_X \omega_a$ can be written in terms of ∇:

$$X^b \nabla_b \omega_a + \omega_b \nabla_a X^b =$$

$$= X^b (\partial_b \omega_a - \Gamma^c{}_{\cancel{ba}} \omega_c) + \omega_b (\partial_a X^b + \Gamma^b{}_{\cancel{ac}} X^c)$$

$$= X^b \partial_b \omega_a + \omega_b \partial_a X^b = \mathcal{L}_X \omega_a$$

Therefore $\mathcal{L}_X \omega_a$ is a $(0,1)$ tensor.

Alternatively we can explicitly show that $\mathcal{L}_X \omega_a$ transforms as a $(0,1)$ under coordinate transformations:

$$\mathcal{L}_X \omega_{a'} = X^{b'} \partial_{b'} \omega_{a'} + \omega_{b'} \partial_{a'} X^{b'}$$

$$= \frac{\partial x^{b'}}{\partial x^c} X^c \frac{\partial x^d}{\partial x^{b'}} \partial_d \left(\frac{\partial x^a}{\partial x^{a'}} \omega_a \right)$$

$$+ \frac{\partial x^b}{\partial x^{b'}} \omega_b \frac{\partial x^a}{\partial x^{a'}} \partial_a \left(\frac{\partial x^{b'}}{\partial x^c} X^c \right)$$

$$= \frac{\partial x^a}{\partial x^{a'}} \left(X^c \partial_c \omega_a + \omega_c \partial_a X^c \right)$$

since $\displaystyle \frac{\partial x^{b'}}{\partial x^c} \frac{\partial x^d}{\partial x^{b'}} \partial_d \left(\frac{\partial x^a}{\partial x^{a'}} \right) = \partial_c \left(\frac{\partial x^a}{\partial x^{a'}} \right) = \partial_{a'} \delta^a{}_c = 0$

$$\frac{\partial x^b}{\partial x^{b'}} \frac{\partial x^a}{\partial x^{a'}} \partial_a \left(\frac{\partial x^{b'}}{\partial x^c} \right) = \frac{\partial x^b}{\partial x^{b'}} \partial_{a'} \left(\frac{\partial x^{b'}}{\partial x^c} \right) = \frac{\partial x^b}{\partial x^{b'}} \partial_c \delta^{b'}{}_{a'} = 0$$

③ $ds^2 = d\rho^2 + \tanh^2\rho \, d\phi^2$

a) $g_{ab} = \mathrm{diag}(1, \tanh^2\rho)$

$g^{ab} = \mathrm{diag}\left(1, \frac{1}{\tanh^2\rho}\right)$

b) Use the Euler-Lagrange equations for the Lagrangian

$L = \dot\rho^2 + \tanh^2\rho \, \dot\phi^2$

ρ) $\frac{\partial L}{\partial \rho} = \frac{2\tanh\rho}{\cosh^2\rho}\dot\phi^2$

$\frac{\partial L}{\partial \dot\rho} = 2\dot\rho$, $\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot\rho}\right) = 2\ddot\rho$

$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot\rho}\right) - \frac{\partial L}{\partial \rho} = 2\ddot\rho - \frac{2\tanh\rho}{\cosh^2\rho}\dot\phi^2 = 0$

$\Rightarrow \ddot\rho - \frac{\tanh\rho}{\cosh^2\rho}\dot\phi^2 = 0 \quad \Rightarrow \Gamma^\rho{}_{\phi\phi} = -\frac{\tanh\rho}{\cosh^2\rho}$

ϕ) $\frac{\partial L}{\partial \phi} = 0$;

$\frac{\partial L}{\partial \dot\phi} = 2\tanh^2\rho\,\dot\phi$; $\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot\phi}\right) =$

$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot\phi}\right) - \frac{\partial L}{\partial \phi} = 2\tanh^2\rho\,\ddot\phi + 4\frac{\tanh\rho}{\cosh^2\rho}\dot\rho\dot\phi = 0$

$\Rightarrow \ddot\phi + 2\frac{1}{\tanh\rho\cosh^2\rho}\dot\rho\dot\phi = 0 \Rightarrow \Gamma^\phi{}_{\rho\phi} = \Gamma^\phi{}_{\phi\rho} = \frac{1}{\sinh\rho\cosh\rho}$