Queen Mary University of London

MTH6132 / MTH6132P: Relativity

Examiners: P. Figueras, R. Russo

The in-class assessed coursework is intended to be completed within **45 minutes**. However, you will have a period of **1 hours** to complete it and submit your solutions.

You should attempt ALL questions. Marks available are shown next to the questions.

The Euler-Lagrange equations governing the geodesics are:

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^a}\right) - \frac{\partial L}{\partial x^a} = 0\,, \quad L = g_{ab}(x)\,\dot{x}^a\dot{x}^b\,, \quad \dot{x}^a = \frac{dx^a}{d\lambda}$$

The Christoffel symbols associated to a metric g_{ab} are computed as

$$\Gamma^a{}_{bc} = \frac{1}{2}g^{ad}(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc})\,.$$

The covariant derivative acting on an arbitrary (k,l) tensor is given by

$$\begin{aligned}\nabla_c T^{a_1\ldots a_k}{}_{b_1\ldots b_l} = {} & \partial_c T^{a_1\ldots a_k}{}_{b_1\ldots b_l} \\ & + \Gamma^{a_1}_{cd}\, T^{da_2\ldots a_k}{}_{b_1\ldots b_l} + \cdots + \Gamma^{a_k}_{cd}\, T^{a_1\ldots a_{k-1}d}{}_{b_1\ldots b_l} \\ & - \Gamma^d{}_{ab_1}\, T^{a_1\ldots a_k}{}_{db_2\ldots b_l} - \cdots - \Gamma^d{}_{ab_l}\, T^{a_1\ldots a_k}{}_{b_1\ldots b_{l-1}d}\,.\end{aligned}$$

Question 1 [10 marks].

(a) Consider an inertial frame S' that is boosted with respect to an inertial frame S by a constant velocity $\mathbf{v}$ along the x^1-axis. Write the Lorentz transformation connecting the coordinates $(x')^a$ of S' and the coordinates x^a of S (with $a = 0, 1, 2, 3$). [**3**]

(b) Write down the Lorentz transformation of the previous point as a 4×4 matrix in terms of the rapidity α defined as $v/c = \tanh(\alpha)$ with $v = |\mathbf{v}|$. Write down the 4×4 matrix relating the dual vectors x'_a and x_a. [**3**]

(c) Write down the expression for the relativistic energy of a particle with speed v and expand it in the regime $v \ll c$ up to order $1/c^2$. Isolate the rest mass energy, the Newtonian kinetic energy and the first relativistic correction. [**4**]

Question 2 [10 marks]. Recall the definition of the Lie derivative of a vector field Y^a along another vector field X^a,

$$\mathcal{L}_X(Y)^a = X^b \partial_b Y^a - Y^b \partial_b X^a$$

Also, recall that the Lie derivative of a scalar ϕ along X^a is just the usual partial derivative:

$$\mathcal{L}_X \phi = X^a \partial_a \phi$$

Given an arbitrary 1-form ω_a,

(a) Using that $\omega_a Y^a$ is a scalar, show that

$$\mathcal{L}_X(\omega)_a = X^b \partial_b \omega_a + \omega_b \partial_a X^b \,.$$

[**5**]

(b) Is $\mathcal{L}_X(\omega)_a$ a (0,1) tensor? [**5**]

Question 3 [10 marks]. Consider the following line element:

$$ds^2 = d\rho^2 + \tanh^2(\rho)\, d\phi^2 \,,$$

with $\rho > 0$ and $\phi \sim \phi + 2\pi$.

(a) Identify the components of the metric tensor g_{ab} and the inverse metric g^{ab}. [**5**]

(b) Compute the Christoffel symbols. [**5**]

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