MTH5131 Actuarial Statistics

Coursework 5 — Solutions

Exercise 1. 1. Assumptions:

(a) The distribution of each X_j depends on a parameter, denoted θ, whose value is fixed (and the same for all the X_j's) but is unknown.

(b) Given θ, the X_j's are independent and identically distributed.

2. $E[s^2(\theta)]$ is estimated by the average of the sample variances:

$$\frac{411.19 + 94.23 + 38.6}{3} = 181.34$$

The sample mean of the $\overline{X_i}$ is

$$\overline{X} = \frac{213.11 + 91.15 + 134.23}{3} = 146.16$$

So $\text{var}[m(\theta)]$ is estimated by

$$\frac{1}{2}\sum_{i=1}^{3}(\overline{X_i} - \overline{X})^2 - \frac{1}{8}E[s^2(\theta)]$$

$$= \frac{(213.11 - 146.16)^2 + (91.15 - 146.16)^2 + (134.23 - 146.16)^2}{2} - \frac{181.34}{8} = 3802.70$$

The credibility factor is then

$$Z = \frac{n}{n + \frac{E[s^2(\theta)]}{\text{var}[m(\theta)]}} = \frac{8}{8 + \frac{181.34}{3802.70}} = 0.994074$$

Exercise 2. 1. $0 \le Z \le 1$

2. The higher the value of Z, the higher is the degree of trust placed in $\overline{X_i}$ compared with $\overline{X}$ as an estimate of next year's expected aggregate claims for risk i.

Exercise 3. 1. $E[s^2(\theta)]$ is estimated by the average of the sample variances:

$$\frac{4,121,280 + 7,299,175 + 3,814,001}{3} = 5,078,152$$

The sample mean of the $\overline{X_i}$ is

$$\overline{X} = \frac{2,517 + 7,814 + 2,920}{3} = 4,417$$

The sample variance of the $\overline{X_i}$ is

$$\frac{1}{3-1}\sum_{i=1}^{3}(\overline{X_i} - \overline{X})^2 = \frac{(2,517 - 4,417)^2 + (7,814 - 4,417)^2 + (2,920 - 4,417)^2}{2} = 8,695,309$$

So $\text{var}[m(\theta)]$ is estimated by

$$\frac{1}{2}\sum_{i=1}^{3}(\overline{X_i} - \overline{X})^2 - \frac{1}{4}E[s^2(\theta)] = 8,695,309 - \frac{1}{4} \times 5,078,152 = 7,425,771$$

The credibility factor is then

$$Z = \frac{n}{n + \frac{E[s^2(\theta)]}{\text{var}[m(\theta)]}} = \frac{4}{4 + \frac{5,078,152}{7,425,771}} = 0.853997$$

2. Z is an increasing function of n, the number of years of past data. If we have more than 4 years of past data, then the credibility factor will increase.

   Z is a decreasing function of $E[s^2(\theta)]$. If $E[s^2(\theta)]$ increases, e.g. if the variance of the claim amounts from one or more of the risks were to increase, then the value of the credibility factor would increase.

   Z is an increasing function of $\text{var}[m(\theta)]$. If $\text{var}[m(\theta)]$ increases,e.g. if there was greater variation between the individual sample means, then Z would increase.

Exercise 4. 1. $\bar{X}_i$ is the average claims for risk i for the 5-year period.
So the missing entry is:

$$\bar{X}_4 = \frac{(44 + 52 + 69 + 55 + 71)}{5} = 58.2$$

We can then calculate the other missing entry:

$$\frac{1}{4}\sum_{j=1}^{5}(X_{ij} - \bar{X}_i)^2 = \frac{1}{4}\left[(44 - 58.2)^2 + \ldots + (71 - 58.2)^2\right] = 132.7$$

To find the estimates we need $\bar{X}$, which is:

$$\bar{X} = \frac{1}{4}\sum_{i=1}^{4}\bar{X}_i = \frac{(50.4 + 68.4 + 74.0 + 58.2)}{4} = 62.75$$

We can then evaluate the estimates directly:

$$E[m(\theta)] \approx \bar{X} = 62.75$$

$$E[s^2(\theta)] \approx \frac{1}{4}\sum_{i=1}^{4}\frac{1}{4}\sum_{j=1}^{5}\left(X_{ij} - \bar{X}_i\right)^2$$

$$= \frac{(39.3 + 17.3 + 215.5 + 132.7)}{4} = 101.2$$

2. We need to find:

$$\text{var}[m(\theta)] \approx \frac{1}{3}\sum_{i=1}^{4}(\bar{X}_i - \bar{X})^2 - \frac{1}{4\times 5}\sum_{i=1}^{4}\frac{1}{4}\sum_{j=1}^{5}\left(X_{ij} - \bar{X}_i\right)^2$$

   The second term is just $\frac{1}{5} \times E[s^2(\theta)]$
   Putting in the required numbers gives us:

$$\text{var}[m(\theta)] \approx \frac{1}{3}\sum_{i=1}^{4}(\bar{X}_i - \bar{X})^2 - \frac{1}{5} \times E[s^2(\theta)]$$

$$= \frac{1}{3}\left[(50.4 - 62.75)^2 + \ldots + (58.2 - 62.75)^2\right] - \frac{1}{5} \times 101.2$$

$$= 90.33$$

3. We can use the estimates we have calculated to find a credibility factor:

$$
\begin{aligned}
Z &= \frac{n}{n + \frac{E[s^2(\theta)]}{\text{var}[m(\theta)]}} \\
&= \frac{5}{5 + \frac{101.2}{90.33}}
\end{aligned}
$$

The credibility factor is the same for each country.

We can then use the basic credibility formula:

$$P = Z\bar{X}_i + (1 - Z)E[m(\theta)]$$

to find the EBCT premiums for each country:

Country 1: $P = 0.8169 \times 50.4 + (1 - 0.8169) \times 62.75 = 52.66$

Country 2: $P = 0.8169 \times 68.4 + (1 - 0.8169) \times 62.75 = 67.37$
Country 3: $P = 0.8169 \times 74.0 + (1 - 0.8169) \times 62.75 = 71.94$
Country 4 $P = 0.8169 \times 58.2 + (1 - 0.8169) \times 62.75 = 59.03$

Exercise 5. We have

$$\overline{P_1} = 163 + 189 + 252 + 199 = 803$$

and similarly

$$\overline{P_2} = 19,353$$

and

$$\overline{P_3} = 71,729$$

For the whole portfolio, we have

$$\overline{P} = 803 + 19,353 + 71,729 = 91,885$$

and

$$
\begin{aligned}
P^* &= \frac{1}{11}\left[803\left(1 - \frac{803}{91,885}\right) + 19,353\left(1 - \frac{19,353}{91,885}\right) + 71,729\left(1 - \frac{71,729}{91,885}\right)\right] \\
&= 2,891.5793
\end{aligned}
$$

The $\overline{X_i}$ can be calculated by

$$\overline{X_1} = \frac{\sum_j Y_{1j}}{P_1} = \frac{14.2 + 15.8 + 22.7 + 19}{803} = 0.089290$$

$$\overline{X_2} = 0.013791$$

$$\overline{X_3} = 0.007654$$

For the whole portfolio, we get

$$\overline{X} = \frac{14.2 + 15.8 + \cdots + 161 + 133}{91,885} = 0.009660$$

Using the summary statistics given in the problem, we can now estimate for $E[m(\theta)]$, $E[s^2(\theta)]$, and $\text{var}[m(\theta)]$:

$$E[m(\theta)] = 0.009660$$

$$E[s^2(\theta)] = \frac{1}{3 \times 3}(0.014667 + 0.006103 + 0.003979) = 0.002750$$

$$\text{var}[m(\theta)] = \frac{1}{2,891.5793}\left[\frac{1}{11}(5.106461 + 0.336408 + 0.292641) - (0.002750)\right] = 0.0001794$$

So the estimated credibility factor for Insurer B is

$$\frac{19,353}{19,353 + \frac{0.002750}{0.0001794}} = 0.999208$$

Hence the credibility premium per unit of risk volume for Insurer B is

$$0.999208 \times 0.013791 + 0.000792 \times 0.009660 = 0.013788$$

Assuming a risk volume in the coming year of $4,800$, the risk premium for Insurer B is £66.18 million.