MTH6127 **Metric Spaces and Topology**

Course work 7 **10 March 2023**

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1. Decide if the mapping $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = \frac{1}{2}\cos x$ is a contraction.

   One has $f'(x) = -\frac{1}{2}\sin x$ and $|f'(x)| \leq \frac{1}{2}$. Thus f satisfies the Lipschitz condition with constant 1/2 and hence is a contraction.

2. Apply the Contraction Mapping Theorem to solve numerically the equation

$$x = \frac{1}{2}\cos x.$$

   By the contraction mapping theorem this equation has a unique solution; it can be found as the limit of iterations with an arbitrary initial condition.

3. Compute the iterations $x_n = f(x_{n-1})$ with two different initial conditions. Compare the results.

   One such computation is shown in the lecture notes.

4. Consider $\mathbb{R}^2$ with the d_1-metric, i.e. $d_1(v, v') = |x - x'| + |y - y'|$ where $v = (x, y)$ and $v' = (x', y')$. Is this metric space complete? Justify your answer.

   This metric is complete, it was shown in lectures.

5. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(v) = (\frac{1}{3}y, \frac{1}{3}(x + 1))$, where $v = (x, y)$. Show that f is a contraction with respect to the d_1-metric.

   If $v = (x, y)$ and $v' = (x', y')$ then

$$f(v) - f(v') = (\frac{1}{3}(y - y'), \frac{1}{3}(x - x'))$$

   and

$$d_1(f(v), f(v')) = \frac{1}{3}|y - y'| + \frac{1}{3}|x - x'| = \frac{1}{3}d_1(v, v').$$

   Thus, f is a contraction.

6. Find the fixed point of f.

Solving the system of equations

$$\begin{cases} \frac{1}{3}y = x, \\ \\ \frac{1}{3}(x+1) = y \end{cases}$$

we find $x = 1/8, \quad y = 3/8.$

7. Assume that a Cauchy sequence (x_n) in a metric space (X, d) contains a subsequence (x_{n_i}) which converges to a point $z \in X$. Show that the whole sequence (x_n) converges to z as well.

Since (x_n) is a Cauchy sequence, the sequence $\sup\{d(x_{n_i}, x_n); n \geq n_i\} = \epsilon_i$ tends to 0, i.e. $\epsilon_i \to 0$ as $i \to \infty$. Thus, for $n \geq n_i$ we have

$$d(x_n, z) \leq d(x_n, x_{n_i}) + d(x_{n_i}, z) \leq \epsilon_i + d(x_{n_i}, z)$$

and when $i \to \infty$ one has $\epsilon_i \to 0$ and $d(x_{n_i}, z) \to 0$ implying our statement.