Queen Mary University of London

MTH793P **Advanced Machine Learning, Semester B, 2023/24**
Coursework 9

In this coursework we will prove a few statements that we used in the lecture.

Robust PCA

1. Let $X \in \mathbb{R}^{m\times n}$, and consider its SVD: $X = U \cdot \Sigma \cdot V^T$. Let $A, B \in \mathbb{R}^{m\times n}$ be two different matrices. Prove that $B = U^T \cdot A \cdot V$ **if and only if** $A = U \cdot B \cdot V^T$.

2. Let $X \in \mathbb{R}^{m\times n}$, and suppose that $\sigma_1, \ldots, \sigma_r$ are the singular values of X ($r = \min(m, n)$). Let $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ be two orthogonal matrices. Define $\tilde{X} = U \cdot X \cdot V^T$. Prove that X and $\tilde{X}$ have the same singular values.

3. Let $X \in \mathrm{R}^{m\times n}$, and recall the definition of the singular thresholding operator

$$D_\tau(X) = US_\tau(\Sigma)V^T,$$

where $U\Sigma V^T$ is the SVD decomposition of X. Prove that:

   (a) $\|D_\tau(X)\|_* \le \|X\|_*$, where $\|\cdot\|_*$ is the nuclear norm.

   (b) $\mathrm{rank}(D_\tau(X)) \le \mathrm{rank}(X)$.

   Under what conditions will we have $\|D_\tau(X)\|_* = \|X\|_*$, $\mathrm{rank}(D_\tau(X)) = \mathrm{rank}(X)$?

Matrix Completion

Let $M \in \mathrm{R}^{m\times n}$. Recall that Ω represents the indexes of known values in M, and $P_\Omega(\cdot)$ is the projection on these locations.

4. Let for any $X, Y \in \mathbb{R}^{m\times n}$ show that $\langle X, P_\Omega(Y)\rangle = \langle P_\Omega(X), Y\rangle$.