Credibility Theory

Credibility is framework in the posterior can often be expressed. the framework has a nice interpretation.

First, we need to review conditional expectation.

$$E[X \mid Y=y] = \sum x \, P(X=x \mid Y=y) \quad \text{if } X, Y \text{ discrete}$$

$$= \int_{-\infty}^{\infty} x \, \frac{f_{X,Y}(x,y)}{f_Y(y)} \, dx \quad \text{if } X, Y \text{ continuous}$$

We let $E[X \mid Y]$ equal $E[X \mid Y=y]$ when $Y=y$.

It is a function of Y.

1. $E[X] = E[E[X|Y]]$
   Law of total probability for expectation

2. $E[f(Y)|Y] = f(Y)$

3. $E[Xf(Y)] = E[E[Xf(Y)|Y]]$ by 1.
   $= E[f(Y)\, E[X|Y]]$

Definition

Two random variables X_1 and X_2 are ~~conditionally~~ conditionally independent given random variable Y if given the value of Y, X_1 and X_2 are independent.

For discrete r.v.'s

$$P(X_1 = x_1, X_2 = x_2 \mid Y = y) = P(X_1 = x_1 \mid Y = y)\, P(X_2 = x_2 \mid Y = y)$$

In particular

$$E[X_1 X_2 \mid Y] = E[X_1 \mid Y]\, E[X_2 \mid Y]$$

Example

Suppose $X_1 \sim \text{Poisson}(Y)$ $\quad Y > 0$

$X_2 \sim \text{Poisson}(2Y)$

are conditionally independent.

Given $Y=2$, $X_1 \sim \text{Poisson}(2)$, $X_2 \sim \text{Poisson}(4)$

independent.

We will show X_1 and X_2 are dependent.

$$E[X_1 X_2] = E\left[E[X_1 X_2 \mid Y]\right]$$

conditional independence

$$= E\left[E[X_1 \mid Y]\, E[X_2 \mid Y]\right]$$

$$= E[Y \cdot 2Y]$$

$$= 2E[Y^2]$$

$$E[X_1] = E\left[E[X_1 \mid Y]\right] = E[Y]$$

$$E[X_2] = E\left[E[X_2 \mid Y]\right] = E[2Y]$$

$$= 2E[Y]$$

$$E[X_1 X_2] = 2E[Y^2] > 2(E[Y])^2$$

$$= E[X_1]\, E[X_2]$$

The inequality holds because for Y not constant.

$\text{Var}(Y) > 0 \Rightarrow E[Y^2] - (E[Y])^2 > 0$

We will introduce credibility theory by example.

Example (Insuring Buses)

A local authority wants to insure a fleet of 10 buses. It wants to insure the fleet for next year. It needs to estimate the average claim amount per bus.

It has two sources of information.

1) Data from this fleet for the past 5 years showing average claim per bus is £160 so total claim should be £1600.
2) Data from bus fleets over the UK for the past 5 years showing average claim per bus is £250 so total claim should be £2500.

The data from 1) is specific to this.

The data from 2) has more sources.

The credibility approach is to take a weighted average of both answers:

$$Z \times 1600 + (1-Z) \times 2500$$

where $0 \le Z \le 1$, Z is called the credibility factor.

Z tells us how much trust we put in the local data.

If $Z=1$, then we put full credibility in the local data.

If $Z=0$, then we put no credibility in the local data

In general, we want to estimate

1) number of claims

or 2) aggregate claim (total number of claims)

due to some risk (policy)

Notation:

- $\bar{X}$ is an estimate based on data from the particular risk
- μ is an estimate based on data from risks similar, but not identicle, to the risk in question.

Our credibility estimate is of the Form

$$Z\bar{X} + (1-Z)\mu$$

What should Z be?

If $\bar{x}$ is based on more years, then Z should be larger.

Bayesian Credibility Theory

- Start with a prior. This is similar to the data contained in μ which is called collateral data.
- Collect data with some likelihood. this is similar to the data contained in $\bar{X}$.
- find the Posterior
- use a loss function and posterior to estimate the parameter.
- ~~Wr~~ Attempt to write the estimate as $Z\bar{x} + (1-Z)\mu$

where $\bar{x}$ comes from the data and μ comes from the prior.

1. Poisson/Gamma

The number of claimes Poisson (λ)

where has prior $\lambda \sim \text{Gamma}(\alpha, \beta)$

The data is the number of claims

for each of the past n years,

$y_1, y_2, \dots, y_n$.

The posterior $\text{Gamma}\left(\alpha + \sum_{i=1}^{n} y_i, \beta + n\right)$

Under quadratic loss, the Bayesian

estimate of λ is

$$\frac{\alpha + \sum_{i=1}^{n} y_i}{\beta + n}$$

We ~~wanto~~ want to put this in the

credibility framework

$\cancel{\bar{X} Z + (1}$

$$Z\bar{X} + (1-Z)\mu$$

where $\bar{X} = \bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i \quad \mu = \frac{\alpha}{\beta}$ (expectation of prior)

Write $\dfrac{\alpha + \sum_{i=1}^{n} y_i}{\beta + n} = Z\bar{y} + (1-Z)\dfrac{\alpha}{\beta}$

where $Z = \dfrac{n}{\beta + n}$

If $n >> \beta$, then $Z \approx 1$

If $n << \beta$, then $Z \approx 0$

As $n \to \infty$, $Z \to 1$

The variance of the prior is $\dfrac{\alpha}{\beta^2}$.

If β is large with respect ~~to α~~ to α, then $\dfrac{\alpha}{\beta^2}$ is small.

When the variance is small, the prior is more credible and Z is smaller.

Normal/Normal Model

We want to estimate pure premium i.e. aggregate claims.

Let X be the pure premium for the coming year.

Assumptions

- $X \sim N(\theta, \sigma_1^2)$ (likelihood)
  where σ_1^2 is known and
  $\theta \sim N(\mu, \sigma_2^2)$ (prior)

  There are three parameters $\sigma_1^2, \mu, \sigma_2^2$
- Let X_j be the pure premium for years $j = 1, \dots, n$ $X_j \sim N(\theta, \sigma_1^2)$ which are dependent, but conditionally independent given θ.
- We observe $X_1, \dots, X_n$ and want to predict X_{n+1}, which the pure premium for year $n+1$.

Remarks

1. Actually, conditional on Θ, the X_j's are i.i.d.

2. The X_j's are dependent, but they are identically distributed, because

$$P(X_i \le x) = \int P(X_i \le x \mid \theta)\, f(\theta)\, d\theta$$

$$= \int \Phi\left(\frac{x-\theta}{\sigma_1}\right) f(\theta)\, d\theta$$

for all i, ~~for all~~ where $f(\theta) = \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{(\theta-\mu)^2}{2\sigma_2^2}}$

3. The X_j's are dependent, because

$$E(X_1 X_2) = E\left(E(X_1 X_2 \mid \theta)\right)$$

$$= E\left(E(X_1 \mid \theta)\, E(X_2 \mid \theta)\right)$$

$$= E(\theta \cdot \theta) = E(\theta^2)$$

$$E[X_1] = E[E(X_1 \mid \theta)] = E[\theta]$$

$$E[X_2] = E[E(X_2 \mid \theta)] = E[\theta]$$

$$E[X_1 X_2] = E[\theta^2] > (E[\theta])^2$$

$$= E[X_1]\, E[X_2].$$

The posterior for θ is

$$N\left(\frac{\mu\sigma_1^2 + n\sigma_2^2\bar{x}}{\sigma_1^2 + n\sigma_2^2}, \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2 + n\sigma_2^2} \right)$$

the Bayesian estimator of θ is

$$\frac{\mu\sigma_1^2 + n\sigma_2^2\bar{x}}{\sigma_1^2 + n\sigma_2^2} \qquad \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

for quadratic, absolute, and all-or-nothing loss.