Since S and T are basis vectors adapted to a coordinate system, their commutator vanishes,

$$[S, T] = 0 \Rightarrow S^b\nabla_b T^a = T^b\nabla_b S^a \,.$$

Keeping this in mind, we can explicitly compute the relative acceleration of geodesics:

$$\begin{aligned}
A^a &= T^b\nabla_b V^a = T^b\nabla_b(T^c\nabla_c S^a)\\
&= T^b\nabla_b(S^c\nabla_c T^a)\\
&= (T^b\nabla_b S^c)(\nabla_c T^a) + T^b\, S^c\nabla_b\nabla_c T^a\\
&= (T^b\nabla_b S^c)(\nabla_c T^a) + T^b\, S^c(\nabla_c\nabla_b T^a + R^a{}_{dbc}\, T^d)\\
&= (T^b\nabla_b S^c)(\nabla_c T^a) + S^c\nabla_c(T^b\nabla_b T^a) - (S^c\nabla_c T^b)\nabla_b T^a + R^a{}_{dbc}\, T^d\, T^b\, S^c\\
&= R^a{}_{dbc}\, T^d\, T^b\, S^c \,.
\end{aligned} \tag{5.7}$$

The first line is just the definition of A^a and the second line comes from $[S, T] = 0$. The third line is just the Leibniz rule; the fourth line replaces a double covariant derivative by derivatives in the opposite order plus the Riemann tensor. The fifth line uses again the Leibniz rule (in the opposite order than usual), and then we cancel two identical terms and notice that the term $T^b\nabla_b T^a$ vanishes because T^a is the tangent vector to a geodesic. The result,

$$A^a = \nabla_T\nabla_T S^a = R^a{}_{dbc}\, T^d\, T^b\, S^c \,, \tag{5.8}$$

is the geodesic deviation equation. It expresses that the relative acceleration between two neighbouring geodesics is proportional to the curvature. Physically the acceleration of neighbouring geodesics is interpreted as a manifestation of the gravitational tidal forces.

5.3 Symmetries of the curvature tensor

In general, a tensor of rank 4 has $4^4 = 256$ components (in spacetime). Symmetries, if present, are important because they reduce the number of independent components. Lowering the index in the definition of the Riemann tensor one obtains

$$R_{abcd} = g_{af}(\partial_c\Gamma^f{}_{bd} - \partial_d\Gamma^f{}_{bc}) + \Gamma_{aec}\Gamma^e{}_{bd} - \Gamma_{aed}\Gamma^e{}_{bc},$$

where

$$R_{abcd} = g_{af}R^f{}_{bcd}, \quad \Gamma_{abd} = g_{af}\Gamma^f{}_{bd}.$$

Now, since R_{abcd} is a tensor, it should have the same symmetries in all frames. Accordingly, choose a *locally inertial frame* for which the Christoffel symbols vanish. For these coordinates one has then that

$$R_{\hat{a}\hat{b}\hat{c}\hat{d}} = g_{\hat{a}\hat{f}}(\partial_{\hat{c}}\Gamma^{\hat{f}}{}_{\hat{b}\hat{d}} - \partial_{\hat{d}}\Gamma^{\hat{f}}{}_{\hat{b}\hat{c}}).$$

where we use hatted indices $\hat{a}, \ldots$ to denote that these expressions are only valid in locally inertial coordinates. Recalling that

$$\Gamma_{abc} = \tfrac{1}{2}\left(\partial_b g_{ca} + \partial_c g_{ba} - \partial_a g_{bc}\right)$$

one obtains

$$R_{\hat{a}\hat{b}\hat{c}\hat{d}} = \tfrac{1}{2}\left(\partial_{\hat{b}}\partial_{\hat{c}}g_{\hat{a}\hat{d}} + \partial_{\hat{a}}\partial_{\hat{d}}g_{\hat{b}\hat{c}} - \partial_{\hat{a}}\partial_{\hat{c}}g_{\hat{b}\hat{d}} - \partial_{\hat{b}}\partial_{\hat{d}}g_{\hat{a}\hat{c}}\right),$$

from where it is easy to read the symmetries of the tensor. It can be checked that

$$R_{abcd} = -R_{bacd}, \quad R_{abcd} = -R_{abdc}, \quad R_{abcd} = R_{cdab}.$$

Furthermore,

$$R_{abcd} + R_{adbc} + R_{acdb} = 0 \quad \Rightarrow \quad R_{a[bcd]} = 0.$$

These symmetries amount to 236 constraints, so R_{abcd} has only 20 non-zero components.

5.4 Bianchi identities, the Ricci and Einstein tensors

Recall that in a locally inertial frame one had that

$$R_{\hat{c}\hat{d}\hat{a}\hat{b}} = \tfrac{1}{2}\left(\partial_{\hat{a}}\partial_{\hat{d}}g_{\hat{c}\hat{b}} - \partial_{\hat{a}}\partial_{\hat{c}}g_{\hat{b}\hat{d}} - \partial_{\hat{b}}\partial_{\hat{d}}g_{\hat{c}\hat{a}} + \partial_{\hat{b}}\partial_{\hat{c}}g_{\hat{a}\hat{d}}\right) .$$

Differentiating with respect to $\hat{x}^e$ one obtains

$$\partial_{\hat{e}}R_{\hat{c}\hat{d}\hat{a}\hat{b}} = \tfrac{1}{2}\partial_{\hat{e}}\left(\partial_{\hat{a}}\partial_{\hat{d}}g_{\hat{c}\hat{b}} - \partial_{\hat{a}}\partial_{\hat{c}}g_{\hat{b}\hat{d}} - \partial_{\hat{b}}\partial_{\hat{d}}g_{\hat{c}\hat{a}} + \partial_{\hat{b}}\partial_{\hat{c}}g_{\hat{a}\hat{d}}\right) .$$

Now consider the sum of the cyclic permutations of the first three indices:

$$\begin{aligned}
&\partial_{\hat{e}}R_{\hat{c}\hat{d}\hat{a}\hat{b}} + \partial_{\hat{c}}R_{\hat{d}\hat{e}\hat{a}\hat{b}} + \partial_{\hat{d}}R_{\hat{e}\hat{c}\hat{a}\hat{b}} \\
&\quad = \tfrac{1}{2}\big(\partial_{\hat{e}}\partial_{\hat{a}}\partial_{\hat{d}}g_{\hat{c}\hat{b}} - \partial_{\hat{e}}\partial_{\hat{a}}\partial_{\hat{c}}g_{\hat{b}\hat{d}} - \partial_{\hat{e}}\partial_{\hat{b}}\partial_{\hat{d}}g_{\hat{c}\hat{a}} + \partial_{\hat{e}}\partial_{\hat{b}}\partial_{\hat{c}}g_{\hat{a}\hat{d}} \\
&\qquad + \partial_{\hat{c}}\partial_{\hat{a}}\partial_{\hat{e}}g_{\hat{d}\hat{b}} - \partial_{\hat{c}}\partial_{\hat{a}}\partial_{\hat{d}}g_{\hat{b}\hat{e}} - \partial_{\hat{c}}\partial_{\hat{b}}\partial_{\hat{e}}g_{\hat{d}\hat{a}} + \partial_{\hat{c}}\partial_{\hat{b}}\partial_{\hat{d}}g_{\hat{a}\hat{e}} \\
&\qquad + \partial_{\hat{d}}\partial_{\hat{a}}\partial_{\hat{c}}g_{\hat{e}\hat{b}} - \partial_{\hat{d}}\partial_{\hat{a}}\partial_{\hat{e}}g_{\hat{b}\hat{c}} - \partial_{\hat{d}}\partial_{\hat{b}}\partial_{\hat{c}}g_{\hat{e}\hat{a}} + \partial_{\hat{d}}\partial_{\hat{b}}\partial_{\hat{e}}g_{\hat{a}\hat{c}}\big) \\
&\quad = 0 .
\end{aligned} \tag{5.9}$$

Since this is an equation between tensors, it is true in any coordinate system, even though we derived it in a particular one. By the antisymmetry property $R_{cdab} = -R_{dcab}$, we can re-write this equation as

$$\nabla_e R_{cdab} + \nabla_d R_{ecab} + \nabla_c R_{deab} = 0 \quad \Rightarrow \quad \nabla_{[e}R_{cd]ab} = 0 . \tag{5.10}$$

This tensorial equation is valid in all frames and is called the *Bianchi identity.* One could have derived it by directly taking the covariant derivative of the Riemann tensor.

The Ricci tensor

The Ricci tensor is obtained by contracting the first and third indices of the Riemann tensor:

$$\begin{aligned}
R_{ab} &\equiv g^{cd}R_{cadb} = R^c{}_{acb} \\
&= \partial_c\Gamma^c{}_{ab} - \partial_a(\Gamma^c{}_{cb}) + \Gamma^d{}_{ab}\Gamma^c{}_{cd} - \Gamma^d{}_{ca}\Gamma^c{}_{db} .
\end{aligned} \tag{5.11}$$

Remark 1. Because of the symmetries of the Riemann tensor one has that the Ricci tensor is symmetric. That is,

$$R_{ab} = R_{ba} .$$

Remark 2. Other contractions of the Riemann tensor vanish or give $\pm R_{ab}$. For example $R^c{}_{cab} = 0$ since R_{cdab} is anti-symmetric in c and d. Also,

$$R^c{}_{abc} = -R^c{}_{acb} = -R_{ab} ,$$

and so on.

Remark 3. One can show that

$$\Gamma^a{}_{ab} = \partial_b \ln\sqrt{|g|} ,$$

where $g = \det(g_{ab})$. Therefore, we have the following formula for the Ricci tensor:

$$R_{ab} = \partial_c\Gamma^c{}_{ab} - \partial_a\partial_b \ln\sqrt{|g|} + \Gamma^c{}_{ab}\partial_c \ln\sqrt{|g|} - \Gamma^d{}_{ca}\Gamma^c{}_{db} . \tag{5.12}$$

The Ricci scalar

The Ricci scalar is defined as the contraction of the indices of the Ricci tensor:

$$R \equiv g^{ab} R_{ab} = g^{ac} g^{bd} R_{abcd}.$$

The Einstein tensor

In the next computations recall that $\nabla_c g_{ab} = 0$ and $\nabla_c g^{ab} = 0$ since the Christoffel connection is metric compatible. Contract twice the Bianchi identity (5.10),

$$\begin{aligned} 0 &= g^{bd} g^{ae} (\nabla_e R_{cdab} + \nabla_c R_{deab} + \nabla_d R_{ecab}) \\ &= \nabla^a R_{ca} - \nabla_c R + \nabla^b R_{cb} \,, \end{aligned} \tag{5.13}$$

or

$$\nabla^a R_{ac} = \frac{1}{2} \nabla_c R \,. \tag{5.14}$$

Note that, unlike the partial derivative, it makes sense to raise an index on the covariant derivative of a tensor because it is another tensor and due to the metric compatibility. We define the *Einstein tensor* as

$$G_{ab} \equiv R_{ab} - \frac{1}{2} R \, g_{ab} \,, \tag{5.15}$$

We then see that the twice-contracted Bianchi identity (5.14) is equivalent to

$$\nabla^a G_{ab} = 0 \,. \tag{5.16}$$

Remark 1. The Einstein tensor, which is symmetric due to the symmetry of the Ricci tensor and the metric, has 10 independent components and it will play a crucial role in general relativity.

Remark 2. By construction, the Einstein tensor is *divergence free.*

The Weyl tensor

The Ricci tensor and the Ricci scalar contain all the information about the possible contractions of the Riemann tensor. The remaining information, namely the trace-free parts, are captured by the Weyl tensor. This tensor is defined as the Riemann tensor with all the contractions removed. In an n-dimensional manifold, one has

$$C_{abcd} = R_{abcd} - \frac{2}{n-2} (g_{a[c} R_{d]b} - g_{b[c} R_{d]a}) + \frac{2}{(n-1)(n-2)} \, g_{a[c} g_{d]b} \, R \,,$$

and hence

$$C^a{}_{bac} = 0 \,.$$

Remark 1. By construction, the Weyl tensor has the same symmetries as the Riemann tensor:

$$C_{abcd} = C_{[ab][cd]} \,, \quad C_{abcd} = C_{cdab} \,, \quad C_{a[bcd]} = 0 \,.$$

Remark 2. The Weyl tensor is only defined in three or more dimensions; in three dimensions it vanishes identically.

Remark 3. A very important property of the Weyl tensor is that it is invariant under conformal transformations of the metric, $g_{ab} \to \Omega(x)^2 \, g_{ab}$, where $\Omega(x)$ is an arbitrary function of the spacetime coordinates.

Example: curvature tensors of the 2-sphere. Consider a round 2-sphere of radius a with metric

$$ds^2 = a^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,.$$

The non-zero Christoffel symbols are given by

$$\begin{aligned} \Gamma^{\theta}{}_{\phi\phi} &= -\sin\theta\,\cos\theta\,, \\ \Gamma^{\phi}{}_{\theta\phi} = \Gamma^{\phi}{}_{\phi\theta} &= \cot\theta\,. \end{aligned}$$

Given the symmetries of the Riemann tensor, the only non-trivial component (up to symmetries) is:

$$\begin{aligned} R^{\theta}{}_{\phi\theta\phi} &= \partial_\theta \Gamma^{\theta}{}_{\phi\phi} - \partial_\phi \Gamma^{\theta}{}_{\theta\phi} + \Gamma^{\theta}{}_{\theta b}\,\Gamma^{b}{}_{\phi\phi} - \Gamma^{\theta}{}_{\phi b}\,\Gamma^{b}{}_{\theta\phi} \\ &= (\sin^{\theta} - \cos^{\theta}) - (0) + (0) - (-\sin\theta\,\cos\theta)(\cot\theta) \\ &= \sin^2\theta\,. \end{aligned}$$

Lowering the first index gives

$$\begin{aligned} R_{\theta\phi\theta\phi} &= g_{\theta c}\, R^{c}{}_{\phi\theta\phi} \\ &= g_{\theta\theta}\, R^{\theta}{}_{\phi\theta\phi} \\ &= a^2\,\sin^2\theta\,. \end{aligned}$$

The Ricci tensor is then computed from $R_{ab} = g^{cd}\, R_{cadb}$, which gives

$$\begin{aligned} R_{\theta\theta} &= g^{\phi\phi}\, R_{\phi\theta\phi\theta} = 1 \\ R_{\theta\phi} &= R_{\phi\theta} = 0 \\ R_{\phi\phi} &= g^{\theta\theta}\, R_{\theta\phi\theta\phi} = \sin^2\theta\,. \end{aligned}$$

Finally, the Ricci scalar is given by,

$$R = g^{ab}\, R_{ab} = g^{\theta\theta} R_{\theta\theta} + g^{\phi\phi} R_{\phi\phi} = \frac{2}{a^2}\,.$$

Note that the scalar of curvature, i.e., the Ricci scalar, decreases as the radius of the sphere increases. In more general cases, we will sometimes refer to the "radius of curvature" of a manifold as providing a length scale over which the curvature varies; the larger the radius of curvature, the smaller the curvature is.

Chapter 6

General Relativity

6.1 Towards the Einstein equations

There are several ways of motivating the Einstein equations. The most natural is perhaps through considerations involving the Equivalence Principle. In gravitational fields there exist local inertial frames in which Special Relativity is recovered. The equation of motion of a free particle in such frames is:

$$\frac{\mathrm{d}^2 x^a}{\mathrm{d}\tau^2} = 0. \tag{6.1}$$

Relative to an arbitrary (accelerating frame) specified by $x'^a = x'^a(x^b)$, the latter becomes:

$$\frac{\mathrm{d}^2 x'^a}{\mathrm{d}\tau^2} + \gamma^a{}_{bc}\frac{\mathrm{d}x'^b}{\mathrm{d}\tau}\frac{\mathrm{d}x'^c}{\mathrm{d}\tau} = 0,$$

where

$$\gamma^a{}_{bc} = \frac{\partial x'^a}{\partial x^d}\frac{\partial^2 x^d}{\partial x'^b \partial x'^c}.$$

Here the $\gamma^a{}_{bc}$ are the "fictitious" terms that arise due to the non-inertial nature of the frame. Now, due to the Equivalence Principle the latter implies that locally gravity is equivalent to acceleration and this in turn gives rise to non-inertial frames. The main idea of General relativity is to argue that gravitation as well as inertial forces should be described by appropriate $\gamma^a{}_{bc}$'s!

Clearly (6.1) is not a tensorial equation since it is not left invariant upon changing frame: although $\frac{dx^a}{d\tau}$ is a well-defined vector, $\frac{d^2x^a}{d\tau^2}$ is not. Note that we can use the chain rule $\frac{d}{d\tau} = \frac{dx^b}{d\tau}\frac{\partial}{\partial x^b}$ to write

$$\frac{\mathrm{d}^2 x^a}{\mathrm{d}\tau^2} = \frac{\mathrm{d}x^b}{\mathrm{d}\tau}\partial_b\left(\frac{\mathrm{d}x^a}{\mathrm{d}\tau}\right)$$

Now it is clear how we can generalise equation (6.1) to curved space: we simply replace the partial derivative by a covariant derivative:

$$\frac{\mathrm{d}x^b}{\mathrm{d}\tau}\partial_b\left(\frac{\mathrm{d}x^a}{\mathrm{d}\tau}\right) \to \frac{\mathrm{d}x^b}{\mathrm{d}\tau}\nabla_b\left(\frac{\mathrm{d}x^a}{\mathrm{d}\tau}\right) = \frac{\mathrm{d}^2 x^a}{\mathrm{d}\tau^2} + \Gamma^a{}_{bc}\frac{\mathrm{d}x^b}{\mathrm{d}\tau}\frac{\mathrm{d}x^c}{\mathrm{d}\tau}.$$

Therefore, we conclude that the generalisation of (6.1) to curved spaces is

$$\frac{\mathrm{d}^2 x^a}{\mathrm{d}\tau^2} + \Gamma^a{}_{bc}\frac{\mathrm{d}x^b}{\mathrm{d}\tau}\frac{\mathrm{d}x^c}{\mathrm{d}\tau} = 0. \tag{6.2}$$

To see that (6.2) indeed describes the motion of test particles in gravitational fields, we can consider the Newtonian limit of this equation. More precisely, in Newtonian limit we assume that particles are moving slowly compared to the speed of light, gravitational fields are weak (so it can be considered as a perturbation of flat space) and that the gravitational field is static. Taking the proper time τ as an affine parameter along the geodesic, "moving slowly" means

$$\frac{\mathrm{d}x^i}{\mathrm{d}\tau} \ll \frac{\mathrm{d}t}{\mathrm{d}\tau}\,.$$

where $i = 1, 2, 3$ denotes the spatial coordinates. In this limit, the geodesic equation (6.2) becomes

$$\frac{\mathrm{d}^2 x^a}{\mathrm{d}\tau^2} + \Gamma^a{}_{00}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 = 0. \tag{6.3}$$

Since the gravitational field is assumed to be static, all t-derivatives of g_{ab} vanish ($\partial_0 g_{ab} = 0$) and the relevant Christoffel symbols simplify

$$\begin{aligned}\Gamma^a{}_{00} &= \tfrac{1}{2} g^{ab}(\partial_0 g_{b0} + \partial_0 g_{0b} - \partial_b g_{00}) \\ &= -\tfrac{1}{2} g^{ab}\partial_b g_{00}\end{aligned} \tag{6.4}$$

Furthermore, since the field is weak, one may adopt a local coordinate system in which

$$g_{ab} = \eta_{ab} + h_{ab}, \quad |h_{ab}| \ll 1. \tag{6.5}$$

From the definition of the inverse metric, $g^{ab}g_{bc} = \delta^a_b$, we find that to first order in h_{ab},

$$g^{ab} = \eta^{ab} - h^{ab}$$

where $h^{ab} = \eta^{ac}\eta^{bd}h_{cd}$. Substituting this into (6.4) and expanding to first order in h_{ab}, one has that

$$\Gamma^a{}_{00} = -\tfrac{1}{2}\eta^{ad}\partial_d h_{00}.$$

Therefore, in this limit the geodesic equation (6.3) becomes:[1]

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}\tau^2} = \frac{1}{2}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 \partial_i h_{00}, \tag{6.6a}$$

$$\frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} = 0, \quad \text{as} \quad \partial_0 h_{00} = 0. \tag{6.6b}$$

From (6.6b) it follows that $\frac{dt}{d\tau}$ is a constant. Also, from

$$\frac{\mathrm{d}x^i}{\mathrm{d}\tau} = \frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau},$$

it follows that

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}\tau^2} = \frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2},$$

which in our case reduces to

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}\tau^2} = \frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2.$$

[1]Note that $\eta^{ij} = \delta^{ij}$ so spatial indices upstairs and downstairs are the same.

Combining the latter with (6.6a) we obtain

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = \frac{1}{2}\partial_i h_{00}. \tag{6.7}$$

The corresponding Newtonian result is

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = -\partial_i \phi \tag{6.8}$$

where ϕ is the gravitational potential. Far from a central body of mass M at a distance r, ϕ is given by

$$\phi = -\frac{GM}{r},$$

where G is Newton's constant of gravitation. Comparing (6.7) and (6.8) one finds that

$$h_{00} = -2\phi + \text{constant}.$$

However, at large distances from M one has that $\phi \to 0$ (gravity becomes negligible) and $g_{ab} \to \eta_{ab}$ (the space becomes flat). Therefore the constant must be zero and we can conclude that

$$h_{00} = -2\phi. \tag{6.9}$$

Substituting in (6.5) one finds

$$g_{00} = -(1 + 2\phi). \tag{6.10}$$

Now, recall that ϕ has dimensions of (velocity)^2, $[\phi] = [GM/R] = L^2/T^2$. Therefore one has that ϕ/c^2 at the surface of the Earth is $\sim 10^{-9}$, on the surface of the Sun is $\sim 10^{-6}$, at the surface of a white dwarf is $\sim 10^{-4}$ while at the surface of a neutron start is $\sim 10^{-2}$. On the other hand, at horizon of a black hole $\phi/c^2 \sim 1$. It follows that in most cases the distortion produced by gravity in the spacetime metric g_{ab} is very small, except near black holes.

We have argued that free particles (subject only to gravitational forces) move along geodesics. In the Newtonian limit of the geodesic equation we have shown how the Christoffel symbols $\Gamma^a{}_{bc}$ are associated with gravitational forces and, in turn, how the spacetime metric g_{ab} can be associated with the gravitational potential. However, we do not know yet what equation the metric g_{ab} has to satisfy. To motivate it, note that the gravitational potential in the Newtonian theory satisfies

$$\nabla^2 \phi = 4\pi G \rho, \tag{6.11}$$

where ρ is the mass density. The relativistic analogue of this equation should be tensorial and of second order in the derivatives of the metric. To take this analogy further, consider two neighbouring particles moving in a gravitational field with a potential ϕ with coordinates $x^i(t)$ and $x^i(t) + \xi^i(t)$ respectively, with $\xi^i(t)$ small and $i = 1, 2, 3$. The equations of motion are then given:

$$\ddot{x}^i = -\frac{\partial \phi(x)}{\partial x^i}$$

and

$$\ddot{x}^i + \ddot{\xi}^i = -\frac{\partial \phi(x)}{\partial x^i} - \xi^j \frac{\partial^2 \phi}{\partial x^i \partial x^j} + O(\xi^2).$$

Subtracting the two last equations:

$$\ddot{\xi}^i = -\xi^j \frac{\partial^2 \phi}{\partial x^i \partial x^j}.$$

This is the relative acceleration of two test particles separated by a 3-vector ξ^i – the second derivative of the potential gives the tidal forces. This is in analogy to the geodesic deviation equation:

$$\nabla_V \nabla_V \xi^a = R^a{}_{cdb} V^c V^d \xi^b,$$

provided that one identifies

$$-\xi^j \frac{\partial^2 \phi}{\partial x^i \partial x^j} \quad \text{and} \quad R^a{}_{cdb} V^c V^d \xi^b.$$

This identification would make clear the relation between gravity and geometry – note that the Riemann tensor involves second derivatives of the metric tensor.

6.2 Principles employed in General Relativity

The main idea underlying General Relativity is that matter –including energy– curves spacetime (assumed to be a smooth Lorentzian manifold). This in turn affects the motion of particles and light rays, postulated to move on timelike and null geodesics of the manifold, respectively. These ideas are understood in conjunction with the main principles of General Relativity, listed below.

(1) Equivalence Principle. In small enough regions of spacetime, the laws of physics reduce to those of Special Relativity; it is impossible to detect the existence of a gravitational field by means of *local* experiments.

(2) Principle of General Covariance. This states that laws of Nature should have the same mathematical form in any reference frame; hence, they should be tensorial.

(3) Principle of minimal gravitational coupling. This is used to derive the General Relativity analogues of Special Relativity results. According to this principle, one should change

$$\eta_{ab} \to g_{ab}, \quad \partial \to \nabla.$$

For example, in Special Relativity the equations for a perfect fluid are given by:

$$T^{ab} = (\rho + p) V^a V^b - p\,\eta^{ab},$$
$$\partial_a T^{ab} = 0.$$

In General Relativity these should be changed to:

$$T^{ab} = (\rho + p) V^a V^b - p\,g^{ab},$$
$$\nabla_a T^{ab} = 0.$$

(4) Correspondence principle. General relativity must agree with Special Relativity in absence of gravitation and with Newtonian gravitational theory in the case of weak gravitational fields and in the non-relativistic limit (slow speed).

6.2.1 The Einstein equations in vacuum

In vacuum, such as in the outside of a body in empty space, one has that the mass density ρ vanishes and the equation for the Newtonian potential becomes:

$$\nabla^2\phi = 0.$$

The Laplace equation involves an object with two indices, namely $\frac{\partial^2\phi}{\partial x^i\partial x^j}$. Therefore, one would guess that the gravitational field equations involve a symmetric geometric object with two indices, and hence the same number of components as the metric g_{ab}, arising from a contraction of the Riemann tensor (since the Riemann has two derivatives of the metric). The Ricci tensor is such a tensor and hence one would be tempted to guess that the gravitational field equations are

$$R_{ab} = 0. \tag{6.12}$$

These are indeed the correct equations for gravity in absence of matter fields and they are known as the *Einstein vacuum field equations.* The equations (6.12) form a set of ten nonlinear, second order partial differential equations for the components of the metric tensor g_{ab}. These are hard to solve, except simple settings with a high degree of symmetry.

Remark 1. One of the simplest solutions to the vacuum equations is the Minkowski metric. Expressing the metric g_{ab} locally as $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$, we see that all the Christoffel symbols vanish, from which $R_{ab} = 0$ is trivially satisfied.

Remark 2. The most general form of the vacuum equations which is tensorial and depends linearly on second derivatives of the metric is:

$$R_{ab} + \Lambda g_{ab} = 0,$$

where Λ is the so-called *Cosmological constant.* Outside Cosmology, Λ is usually taken to be zero.

6.2.2 The (full) Einstein Equations

Matter in relativity is described by a (0,2) tensor T_{ab} called the energy-momentum tensor. Therefore, in the presence of matter, one would be tempted to generalise (6.12) to

$$R_{ab} = \kappa\, T_{ab}$$

for some coupling constant κ. In fact, Einstein did suggest this equation. However, this equation is problematic for the following reason. The mass-energy is conserved and this is described by $\nabla^a T_{ab} = 0$, consistent with the minimal coupling principle that generalises of the equations motion in the Special Relativity case. However, in general $\nabla^a R_{ab} \neq 0$. Therefore, consistency with the conservation of mass-energy implies that we have to equate T_{ab} with a curvature tensor with vanishing divergence. There is only one (0,2) tensor, constructed from the Ricci tensor, which is automatically conserved: the Einstein tensor

$$G_{ab} = R_{ab} - \frac{1}{2} R\, g_{ab}$$

which always satisfies $\nabla^a G_{ab} = 0$. Therefore, one is led to propose

$$G_{ab} = \kappa\, T_{ab}.$$

as the field equation for the spacetime metric g_{ab} in the presence of matter-energy sources. Note, however, that since $\nabla^a g_{ab} = 0$, we could also have written

$$G_{ab} + \Lambda g_{ab} = \kappa T_{ab}. \tag{6.13}$$

These are the complete Einstein field equations for the metric g_{ab} of a spacetime.

Note that the Einstein equations are the simplest compatible with the Equivalence Principle, but they are not the only ones. In general, the Einstein field equations are extremely complicated set of non-linear partial differential equations. In some simple settings, analytic solutions may be found. These include:

(i) The vacuum spherically symmetric static case (the Schwarzschild spacetime).

(ii) The weak field case (gravitational waves).

(iii) The isotropic and homogeneous case (Cosmology).

We will study cases (i) and (ii) in the following sections.

6.2.3 Newtonian limit

To determine the value of the constant κ one needs to make contact with the Newtonian theory. In this subsection we are going to see how (6.13) (with $\Lambda = 0$) reproduces the Poisson equation for the gravitational potential in the Newtonian limit. From now on, we will set $\Lambda = 0$ unless otherwise stated.

Contracting both sides of (6.13) we find $R = -\kappa T$, which allows us to rewrite (6.13) as

$$R_{ab} = \kappa \left(T_{ab} - \tfrac{1}{2} T \, g_{ab} \right) \tag{6.14}$$

We want to show that this equation reduces to Newtonian gravity in the weak-field, time-independent and slowly moving limit. For simplicity, we consider dust as the source of energy-momentum, for which

$$T_{ab} = \rho \, U_a \, U_b \, ,$$

where U^a is the dust four-velocity, and ρ is the energy density in the rest frame. The "dust" we are considering is a massive body, such as the Sun. Without loss of generality, we can work in the dust rest frame, in which

$$U^a = (U^0, 0, 0, 0) \, .$$

We can fix U^0 using the normalisation condition $g_{ab} U^a U^b = -1$. In the weak field limit, from (6.9) and (6.10) we can write

$$g_{00} = -1 + h_{00} \, , \quad g^{00} = -1 - h_{00} \, . \tag{6.15}$$

Then, to first order in h_{ab} we get

$$U^0 = 1 + \tfrac{1}{2} \, h_{00} \, .$$

In fact, we are already assuming that ρ is small. Therefore, the contributions from h_{00} to T_{ab} coming from the U_0 terms will be of higher order, and we can simply take $U^0 = 1$, and correspondingly $U_0 = -1$. Then,

$$T_{00} = \rho \, ,$$

and all the other components of the stress-energy tensor T_{ab} vanish. In this limit, the rest energy $\rho = T_{00}$ will be much larger than the other terms in T_{ab}, so we can focus on the $a = b = 0$ component of (6.14). To the lowest non-trivial order, the trace of the energy momentum tensor is

$$T = g^{ab} T_{ab} = g^{00} T_{00} = -T_{00} = -\rho\,.$$

and hence, the 00-component of (6.14) becomes

$$R_{00} = \tfrac{1}{2}\,\kappa\,\rho\,. \tag{6.16}$$

Now we need to express the lhs of this equation in terms of the metric. To do so, we have to compute $R_{00} = R^a{}_{0a0} = R^i{}_{0i0}$. We have

$$R^i{}_{0j0} = \partial_j \Gamma^i{}_{00} - \partial_0 \Gamma^i{}_{j0} + \Gamma^i{}_{ja} \Gamma^a{}_{00} - \Gamma^i{}_{0a} \Gamma^a{}_{j0}\,.$$

Note that the second term in this expression is a time derivative, which vanishes for static fields. The third and fourth terms are of the form $(\Gamma)^2$, and since the Christoffels Γ are of first order in the metric perturbation h_{ab}, these terms are of higher order and can be neglected. Therefore, to first order in h_{ab} we have $R^i{}_{0j0} = \partial_j \Gamma^i{}_{00}$. From this, we compute

$$\begin{aligned}
R_{00} &= R^i{}_{0i0} \\
&= \partial_i \left[\tfrac{1}{2} g^{ia} \left(\partial_0 g_{a0} + \partial_0 g_{0a} - \partial_a g_{00}\right)\right] \\
&= -\,\tfrac{1}{2}\,\delta^{ij} \partial_i \partial_j h_{00} \\
&= -\,\tfrac{1}{2} \nabla^2 h_{00}\,.
\end{aligned}$$

Then, equation (6.16) becomes

$$\nabla^2 h_{00} = -\kappa\,\rho\,. \tag{6.17}$$

From equation (6.9) we have $h_{00} = -2\,\phi$. Comparing with the Poisson equation for Newtonian gravity (6.11), we see that General Relativity does indeed reproduce Newtonian gravity if we set $\kappa = 8\pi\, G$, where G is Newton's gravitational constant.

Having fixed the normalisation correctly to reproduce the Newtonian limit we arrive at the final form of Einstein's equations for general relativity:

$$R_{ab} - \tfrac{1}{2}\, R\, g_{ab} = 8\pi\, G\, T_{ab}\,. \tag{6.18}$$

6.3 The Schwarzschild solution

In GR, the unique spherically symmetric vacuum solution of the Einstein equations is the Schwarzschild metric. It is second in importance only to Minkowski space and it corresponds to the static, spherically symmetric gravitational field in empty space surrounding a (spherically symmetric) source, such as a star. As we shall see later, it also represents a black hole.

The assumption of spherical symmetry and staticity severely constraints the form of the line element. Firstly, assuming that the spacetime is static means that there exists a timelike Killing vector field such that, far away from any sources, is of the form ∂_t, which is the canonical timelike Killing vector field in Minkowski space. Furthermore, in these coordinates the line element is invariant under a time inversion $t \to -t$. The assumption to preserve spherical symmetry implies that coordinates can be chosen such that the line element possesses an explicit round sphere, $d\Omega^2_{(2)} = d\theta^2 + \sin^2\theta\, d\phi^2$, where (θ, ϕ) are the standard angular coordinates on a unit 2-sphere. Therefore, with our symmetry assumptions, the most general line element that we can write down is of the following form:

$$\mathrm{d}s^2 = -e^{2A(r)}\mathrm{d}t^2 + e^{2B(r)}\mathrm{d}r^2 + r^2 e^{2C(r)}(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,. \tag{6.19}$$

We can use our freedom to choose the coordinates to simplify (6.19) further. Defining a new radial coordinate,

$$\bar{r} = r\, e^{C(r)} \quad \Rightarrow \quad d\bar{r} = \left(1 + r\,\tfrac{dC}{dr}\right) e^{C(r)}\, dr\,, \tag{6.20}$$

the metric (6.19) becomes

$$\mathrm{d}s^2 = -e^{2A(r)}\mathrm{d}t^2 + \left(1 + r\,\tfrac{dC}{dr}\right)^{-2} e^{2(B(r)-C(r))}\mathrm{d}\bar{r}^2 + \bar{r}^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,, \tag{6.21}$$

where r is now a function of $\bar{r}$ defined by (6.20). Making the following relabelings,

$$\bar{r} \to r\,, \quad \left(1 + r\,\tfrac{dC}{dr}\right)^{-2} e^{2(B(r)-C(r))} \to e^{2B(r)}\,,$$

the metric (6.21) becomes

$$\mathrm{d}s^2 = -e^{2A(r)}\mathrm{d}t^2 + e^{2B(r)}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,. \tag{6.22}$$

This is the most general static and spherically symmetric spacetime. Note that in these coordinates, r has a physical meaning, namely is the areal radius of the 2-spheres.

Given the form of the metric (6.22), we are now ready to solve the Einstein vacuum equations,

$$R_{ab} = 0\,,$$

From (6.22), we find that the only non-vanishing components of the Ricci tensor are:

$$R_{tt} = e^{2(A-B)}\left(A'' + A'^2 - A'\,B' + \tfrac{2}{r}\,A'\right)\,, \tag{6.23}$$

$$R_{rr} = -A'' - A'^2 + A'\,B' + \tfrac{2}{r}\,B'\,, \tag{6.24}$$

$$R_{\theta\theta} = e^{-2B}\left[r(B' - A') - 1\right] + 1\,, \tag{6.25}$$

$$R_{\phi\phi} = \sin^2\theta\, R_{\theta\theta}\,, \tag{6.26}$$

where $'$ denotes $\frac{d}{dr}$. Having calculated the components of the Ricci tensor, we now have to equate them to zero. Since all components have to vanish independently, we can consider the combination

$$0 = e^{2(B-A)}R_{tt} + R_{rr} = \frac{2}{r}(A' + B')\,,$$

which implies $A(r) = -B(r) + c$, where c is a constant. We can set this constant to zero by rescaling the time coordinate by $t \to e^{-c}\,t$, after which we have

$$A(r) = -B(r)\,. \tag{6.27}$$

Considering $R_{\theta\theta} = 0$, using the previous result this equation now becomes

$$e^{2A}(2\,r\,A' + 1) = 1\,,$$

which is equivalent to

$$\partial_r\left(r\,e^{2A}\right) = 1\,.$$

This equation can be straightforwardly integrated to obtain

$$e^{2A(r)} = 1 - \frac{R_S}{r}\,, \tag{6.28}$$

where R_S is an undetermined constant. Using the results (6.27) and (6.28), we find that the spacetime metric that solves the Einstein vacuum equations is

$$\mathrm{d}s^2 = -\left(1 - \frac{R_S}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-\frac{R_S}{r}} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\,. \tag{6.29}$$

This metric depends on a single parameter, namely the constant R_S, which is called the Schwarzschild radius. To fix this constant in terms of a physical parameter, recall that in the weak field limit (i.e., far away from the source), the tt-component of the spacetime metric sourced by a mass M is given by

$$g_{tt} = -\left(1 - \frac{2GM}{r}\right)\,. \tag{6.30}$$

The metric (6.29) should reduce to the weak field case when $r \gg R_S$, but for the tt-component to agree with (6.30) we need to identify

$$R_S = 2GM\,.$$

We can now write down the final form of a static, spherically symmetric spacetime metric

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\,. \tag{6.31}$$

This line element is known as the *Schwarzschild metric*, and it depends on a single parameter, namely M. This parameter can be interpreted as the mass of the spacetime. Note that as $M \to 0$, we recover Minkowski space, as expected. Note also that as $r \to \infty$, the metric (6.31) becomes more like Minkowski space; this property is known as asymptotic flatness.

Remark 1. This solution demonstrates how the presence of mass curves flat spacetime.

Remark 2. The solution only applies to the exterior of a star, where there is vacuum. We will see shortly that, in the absence of matter, this solution describes a black hole.

Remark 3. *The Birkhoff Theorem:* a spherically symmetric solution in vacuum is necessarily static. That is, there is no time dependence is spherically symmetric solutions. Therefore, the assumption of staticity is not necessary.

Singularities

We see from (6.31) that the some of the metric coefficients become infinite or zero at $r = 0$ and $r = 2GM$, which suggests that something may be going wrong there. The metric coefficients are of course coordinate dependent; hence, it is entirely possible that the apparent problems at those values of the radial coordinate r are simply coordinate singularities that result in a breakdown of the coordinates rather than a problem with the spacetime manifold itself. For instance, this is precisely what happens at the origin of polar coordinates in flat space, where the metric $ds^2 = dr^2 + r^2 d\theta^2$ becomes degenerate and $g^{\theta\theta}$ blows up at $r = 0$. Of course, we know that there is nothing wrong with flat space at $r = 0$: this point is equivalent to any other point of the manifold, and by changing to Cartesian coordinates we see that both the metric $ds^2 = dx^2 + dy^2$ and its inverse are perfectly well-behaved at $x = y = 0$ ($r = 0$).

Therefore, in GR we need to assess singularities in a coordinate independent way. In general, this is difficult but for our present purposes we will identify singularities as places where the curvature of spacetime becomes infinite. The curvature is measured by the Riemann tensor, so to say that the curvature become infinite one cannot simply use the components of this tensor since they are coordinate-dependent. However, from the curvature one can construct scalars and, since the latter are coordinate independent, they provide a meaningful way to assess when the curvature becomes infinite. Scalars involving the Ricci scalar R or the Ricci tensor, e.g., $R_{ab}R^{ab}$, are not useful since they are fixed by the Einstein equations and, in the vacuum case, they trivially vanish. On the other hand, scalar quantities such as $R_{abcd}R^{abcd}$ or $R_{abcd}R^{cdef}R_{ef}{}^{ab}$ contain information about the curvature of the spacetime which is not determined by the Einstein equations and hence we can use them to detect physical singularities. If any of these scalars (but not necessarily all of them) blows up as we approach a certain point on the manifold, we regard that point as a singularity of the curvature. We should also check that this point is not infinitely far away in physical distance, that is, that it can be reached by observers or light travelling a finite distance along a curve.

Therefore, we have a sufficient condition for a point to be considered a singularity, but it is not a necessary condition. For the Schwarzschild metric (6.31), we find that

$$R_{abcd}R^{abcd} = \frac{48G^2M^2}{r^6}\,. \tag{6.32}$$

This scalar of the curvature, known as the Kretschmann scalar, blows up at $r = 0$, which is sufficient to convince us that $r = 0$ is a true singularity in the manifold. The other potentially troublesome point is $r = 2GM$, the Schwarzschild radius. We see that the Kretschmann scalar (6.32) (and in fact any other curvature scalar) is perfectly well-behaved there. This suggests that the singularity at $r = 2GM$ may just be a coordinate singularity and that the spacetime metric may be perfectly smooth there in more appropriate coordinates. We will see that this is indeed the case and that it is possible to find coordinates such that the Schwarzschild metric is smooth at $r = 2GM$; as we shall see, this surface corresponds to the event horizon of a black hole.

In the case of the Sun, it is a body that extends to a radius of $R_\odot = 10^6 GM_\odot$. Therefore, the surface $r = 2GM_\odot$ is far inside the Sun and hence the Schwarzschild metric does not apply there. On the other hand, there are compact objects for which the Schwarzschild metric is valid everywhere; as we will see, these objects are in fact black holes.

Remark 4. Uniqueness Theorem (Israel '67): The Schwarzschild metric (6.31) is the unique static, topologically spherical, asymptotically flat black hole solution of the Einstein vacuum equations.

6.4 Geodesics of the Schwarzschild geometry

The classical experimental tests of General Relativity are based on the Schwarzschild solution. These are based on the comparison of the trajectories of freely falling particles and light rays in gravitational field of a central body with their counterparts in Newtonian theory. Therefore, we have to consider geodesics, both timelike and null, in the