WEEK 8

Symmetries of the Riemann tensor

Consider

$$R_{abcd} = g_{ae} R^e{}_{bcd} =$$
$$= g_{ae}\left(\partial_c \Gamma^e{}_{bd} - \partial_d \Gamma^e{}_{bc}\right) + \Gamma_{aec}\Gamma^e{}_{bd} - \Gamma_{aed}\Gamma^e{}_{bc}$$

where $\Gamma_{abd} = g_{a\delta}\Gamma^\delta{}_{bd} = \frac{1}{2}\left(\partial_b g_{da} + \partial_d g_{ba} - \partial_a g_{bd}\right)$

Since R_{abcd} is a tensor, it has the same symmetries in all coordinate frames. Consider a locally inertial frame, i.e., $g_{\hat{a}\hat{b}} = \mathrm{diag}(-1, 1 \ldots 1)$ and $\partial_{\hat{c}} g_{\hat{a}\hat{b}}\big|_p = 0 \Rightarrow \Gamma^{\hat{a}}{}_{\hat{b}\hat{c}} = 0$,

$$\Rightarrow R_{\hat{a}\hat{b}\hat{c}\hat{d}} = g_{\hat{a}\hat{\delta}}\left(\partial_{\hat{c}}\Gamma^{\hat{\delta}}{}_{\hat{b}\hat{d}} - \partial_{\hat{d}}\Gamma^{\hat{\delta}}{}_{\hat{b}\hat{c}}\right)$$
$$= \frac{1}{2}\left(\partial_{\hat{b}}\partial_{\hat{c}} g_{\hat{a}\hat{d}} + \partial_{\hat{a}}\partial_{\hat{d}} g_{\hat{b}\hat{c}} - \partial_{\hat{a}}\partial_{\hat{c}} g_{\hat{b}\hat{d}} - \partial_{\hat{b}}\partial_{\hat{d}} g_{\hat{a}\hat{c}}\right)$$

We can now easily read off the symmetries:

$R_{abcd} = -R_{bacd}$, $R_{abcd} = -R_{abdc}$, $R_{abcd} = R_{cdab}$

$R_{abcd} + R_{adbc} + R_{acdb} = R_{a[dbc]} = 0$

(1st Bianchi identity)

$\Rightarrow$ In 4d, R_{abcd} has 20 independent components

Bianchi identity

Recall that in a locally inertial frame,

$$R_{\hat{c}\hat{d}\hat{a}\hat{b}} = \frac{1}{2}\left(\partial_{\hat{a}}\partial_{\hat{d}}g_{\hat{c}\hat{b}} - \partial_{\hat{a}}\partial_{\hat{c}}g_{\hat{b}\hat{d}} - \partial_{\hat{b}}\partial_{\hat{d}}g_{\hat{c}\hat{a}} + \partial_{\hat{b}}\partial_{\hat{c}}g_{\hat{a}\hat{d}}\right)$$

ally

$$\rightarrow \partial_{\hat{e}}R_{\hat{c}\hat{d}\hat{a}\hat{b}} = \frac{1}{2}\partial_{\hat{e}}\left(\partial_{\hat{a}}\partial_{\hat{d}}g_{\hat{c}\hat{b}} - \partial_{\hat{a}}\partial_{\hat{c}}g_{\hat{b}\hat{d}} - \partial_{\hat{b}}\partial_{\hat{d}}g_{\hat{c}\hat{a}} + \partial_{\hat{b}}\partial_{\hat{c}}g_{\hat{a}\hat{d}}\right)$$

Now consider the sum of the cyclic permutations of the first three indices:

$$\partial_{\hat{e}}R_{\hat{c}\hat{d}\hat{a}\hat{b}} + \partial_{\hat{c}}R_{\hat{d}\hat{e}\hat{a}\hat{b}} + \partial_{\hat{d}}R_{\hat{e}\hat{c}\hat{a}\hat{b}} =$$

$$\begin{aligned} = \frac{1}{2}\Big(& \partial_{\hat{e}}\partial_{\hat{a}}\partial_{\hat{d}}g_{\hat{c}\hat{b}} - \partial_{\hat{e}}\partial_{\hat{a}}\partial_{\hat{c}}g_{\hat{b}\hat{d}} - \partial_{\hat{e}}\partial_{\hat{b}}\partial_{\hat{d}}g_{\hat{c}\hat{a}} + \partial_{\hat{e}}\partial_{\hat{b}}\partial_{\hat{c}}g_{\hat{a}\hat{d}} \\ & + \partial_{\hat{c}}\partial_{\hat{a}}\partial_{\hat{e}}g_{\hat{d}\hat{b}} - \partial_{\hat{c}}\partial_{\hat{a}}\partial_{\hat{d}}g_{\hat{b}\hat{e}} - \partial_{\hat{c}}\partial_{\hat{b}}\partial_{\hat{e}}g_{\hat{d}\hat{a}} + \partial_{\hat{c}}\partial_{\hat{b}}\partial_{\hat{d}}g_{\hat{a}\hat{e}} \\ & + \partial_{\hat{d}}\partial_{\hat{a}}\partial_{\hat{c}}g_{\hat{e}\hat{b}} - \partial_{\hat{d}}\partial_{\hat{a}}\partial_{\hat{e}}g_{\hat{b}\hat{c}} - \partial_{\hat{d}}\partial_{\hat{b}}\partial_{\hat{c}}g_{\hat{e}\hat{a}} + \partial_{\hat{d}}\partial_{\hat{b}}\partial_{\hat{e}}g_{\hat{a}\hat{c}}\Big) \end{aligned}$$

$$= 0$$

This is a tensor equation and hence it should be true in any coordinate system:

$$\nabla_e R_{cdab} + \nabla_d R_{ecab} + \nabla_c R_{deab} = \nabla_{[e}R_{cd]ab} = 0$$

$\rightarrow$ 2nd Bianchi identity

· The Ricci tensor

$$R_{ab} = g^{cd} R_{cadb} = \partial_c \Gamma^c{}_{ab} - \partial_a \Gamma^c{}_{cb} + \Gamma^d{}_{ab} \Gamma^c{}_{cd} - \Gamma^d{}_{ca} \Gamma^c{}_{db}$$

Rmk: $R_{ab} = R_{ba}$

Note that $\Gamma^a{}_{ab} = \partial_b \ln\sqrt{|g|}$ where $g = \det g_{ab}$. Then

$$R_{ab} = \partial_c \Gamma^c{}_{ab} - \partial_a \partial_b \ln\sqrt{|g|} + \Gamma^d{}_{ab} \partial_d \ln\sqrt{|g|} - \Gamma^d{}_{ca} \Gamma^c{}_{db}$$

· The Ricci Scalar : $R = g^{ab} R_{ab} = g^{ac} g^{bd} R_{abcd}$

· The Einstein tensor : $G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}$

· $\nabla^a G_{ab} = 0$

· Proof: Contract twice the 2nd Bianchi identity

$$\begin{aligned} 0 &= g^{bd} g^{ae} \left(\nabla_e R_{cdab} + \nabla_c R_{deab} + \nabla_d R_{ecab} \right) \\ &= \nabla^a R_{ca} + \nabla_c R + \nabla^b R_{cb} \\ &= 2 \left(\nabla^a R_{ac} + \frac{1}{2} \nabla_c R \right) = 2 \nabla^a G_{ac} \end{aligned}$$

→ The fact that G_{ab} is divergence free is a geometric property!

- ## The Weyl tensor

The Ricci tensor and the Ricci scalar contain all the information about the contractions of the Riemann tensor. The Weyl tensor is the trace-free part of the Riemann:

$$C_{abcd} = R_{abcd} - \frac{2}{n-2}\left(g_{a[c}R_{d]b} - g_{b[c}R_{d]a}\right) + \frac{2}{(n-2)(n-1)} R\, g_{a[c}\, g_{d]b}$$

- $C^{a}{}_{bac} = 0$
- The Weyl tensor has the same symmetries as the Riemann:

$$C_{abcd} = C_{[ab][cd]}, \quad C_{abcd} = C_{cdab}, \quad C_{a[bcd]} = 0$$

- The Weyl tensor is invariant under conformal transformations of the metric: $g_{ab} \to \Omega(x)^2 g_{ab}$

GENERAL RELATIVITY

· Towards the Einstein equations

Recall the Equivence Principle: in gravitational fields, there exist inertial frames in which Special Relativity applies. The equation of motion of a free particle in such frames is:

$$\frac{d^2x^a}{d\tau^2} = 0$$

Relative to an accelerating frame $x'^a = x'^a(x^b)$

$$\frac{dx^a}{d\tau} = \frac{\partial x^a}{\partial x'^b}\frac{dx'^b}{d\tau}$$

$$\frac{d^2x^a}{d\tau^2} = \frac{\partial x^a}{\partial x'^b}\frac{d^2x'^b}{d\tau^2} + \frac{\partial^2 x^a}{\partial x'^c \partial x'^b}\frac{dx'^c}{d\tau}\frac{dx'^b}{d\tau} = 0$$

$$\frac{\partial x'^c}{\partial x^a} \Rightarrow \quad \frac{d^2x'^a}{d\tau^2} + \frac{\partial x'^a}{\partial x^d}\frac{\partial^2 x^d}{\partial x'^b \partial x'^c}\frac{dx'^b}{d\tau}\frac{dx'^c}{d\tau} =$$

$$= \frac{d^2x'^a}{d\tau^2} + \gamma^a{}_{bc}\frac{dx'^b}{d\tau}\frac{dx'^c}{d\tau} = 0$$

$\gamma^a{}_{bc}$: "fictitious" force terms that arise due to the non-inertial nature of frame

Equivalence Ppl: locally gravity = acceleration and acceleration gives rise to non-inertial frames

GR: gravity and acceleration are described by appropriate Γ's.

The problem with the equation of motion for a free particle above is that it's not tensorial:

$$\frac{d^2x^a}{d\tau^2} = \frac{dx^b}{d\tau}\partial_b\left(\frac{dx^a}{d\tau}\right)$$

To get a tensorial equation we replace $\partial_b \to \nabla_b$:

$$\rightarrow \quad \frac{dx^b}{d\tau}\nabla_b\left(\frac{dx^a}{d\tau}\right) = \frac{dx^b}{d\tau}\partial_b\left(\frac{dx^a}{d\tau}\right) + \Gamma^a{}_{bc}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} =$$

$$= \frac{d^2x^a}{d\tau^2} + \Gamma^a{}_{bc}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = 0$$

$\Rightarrow$ The geodesic equation

<u>Conclusion</u>: The geodesic equation describes the motion of test particles in gravitational fields

- <u>Newtonian limit</u>:
  - Small velocities compared to the speed of light
  - Weak gravitational fields
  - Static gravitational field

If τ is the affine parameter along the geodesic, moving slowly means

$$\frac{dx^i}{d\tau} \ll \frac{dt}{d\tau}$$

$$\Rightarrow \quad \frac{d^2x^a}{d\tau^2} + \Gamma^a{}_{bc}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} \approx \frac{d^2x^a}{d\tau^2} + \Gamma^a{}_{tt}\left(\frac{dt}{d\tau}\right)^2 = 0$$

- Static gravitational field : $\partial_t g_{ab} = 0$

$$\Rightarrow \quad \Gamma^a{}_{tt} = \frac{1}{2} g^{ab}\left(\partial_t g_{tb} + \partial_t g_{tb} - \partial_b g_{tt}\right) = -\frac{1}{2} g^{ab}\partial_b g_{tt}$$
$$= -\frac{1}{2} g^{ai}\partial_i g_{tt}$$

- Weak gravitational field : $g_{ab} = \eta_{ab} + h_{ab}$, $|h_{ab}| \ll 1$

$\Rightarrow$ Since $g^{ac}g_{cb} = \delta^a{}_b \rightarrow g^{ab} = \eta^{ab} - h^{ab} + O(h^2)$

where $h^{ab} = \eta^{ac}\eta^{bd}h_{cd}$

$$\Rightarrow \quad \Gamma^a{}_{tt} = -\frac{1}{2}\delta^{ai}\partial_i h_{tt}$$

Therefore, in this limit the geodesic equation becomes :

$$\frac{d^2t}{d\tau^2} = 0 \quad \Rightarrow \quad \frac{dt}{d\tau} = \text{constant}$$

$$\frac{d^2x^i}{d\tau^2} = \frac{1}{2}\left(\partial_i h_{tt}\right)\left(\frac{dt}{d\tau}\right)^2$$

Note: $\frac{dx^i}{d\tau} = \frac{dx^i}{dt}\frac{dt}{d\tau} \Rightarrow \frac{d^2x^i}{d\tau^2} = \frac{d^2x^i}{dt^2}\left(\frac{dt}{d\tau}\right)^2 + \frac{dx^i}{dt}\cancelto{0}{\frac{d^2t}{d\tau^2}}$

$$\Rightarrow \frac{d^2x^i}{d\tau^2} = \frac{d^2x^i}{dt^2}\cancel{\left(\frac{dt}{d\tau}\right)^2} = \frac{1}{2}(\partial_i h_{tt})\cancel{\left(\frac{dt}{d\tau}\right)^2}$$

$$\Rightarrow \frac{d^2x^i}{dt^2} = \frac{1}{2}(\partial_i h_{tt})$$

The Newtonian equation for a particle moving in a gravitation field is

$$\frac{d^2x^i}{dt^2} = -\partial_i\phi \quad \text{with} \quad \phi = -\frac{GM}{r}$$

ϕ: gravitational potential for a central body of mass M at a distance r from the particle

Comparing we find: $h_{tt} = -2\phi + \text{constant}$

At large distances from the mass M, $\phi \to 0$ and gravity should become negligible: $g_{ab} \to \eta_{ab}$

$$\Rightarrow h_{tt} = -2\phi$$

$$\Rightarrow g_{tt} = -(1+2\phi)$$

ϕ @ the surface of the Earth $\sim 10^{-9}$

ϕ " " Sun $\sim 10^{-6}$

ϕ " " white dwarf $\sim 10^{-4}$

ϕ " " neutron star $\sim 10^{-2}$

ϕ " " black hole ~ 1

To motivate the equation for the metric recall that in Newtonian gravity ϕ is given by

$$\nabla^2 \phi = 4\pi G \rho \qquad \rho: \text{ mass density}$$

→ The relativistic analogue of this equation should be tensorial and involve 2nd derivatives of the metric since $g_{ab} \sim \phi$

Consider the motion of two neighbouring particles located at $x^i(t)$ and $x^i(t) + \xi^i(t)$ in a Newtonian gravitation field ϕ:

$$\frac{d^2x^i}{dt^2} = -\partial_i \phi(x)$$

$$\frac{d^2}{dt^2}\left(x^i(t) + \xi^i(t) \right) = \frac{d^2x^i}{dt^2} + \frac{d^2\xi^i}{dt^2} = -\partial_i \phi(x+\xi) \approx -\partial_i\phi - \xi^j \partial_i \partial_j \phi + O(\xi^2)$$

$$\Rightarrow \quad \frac{d^2 \xi^i}{dt^2} = - \xi^j \partial_i \partial_j \phi$$

Compare with the geodesic deviation equation:

$$\nabla_V \nabla_V \xi^a = R^a{}_{cdb} V^c V^d \xi^b$$

$$\Rightarrow \quad \partial_i \partial_j \phi \sim R^a{}_{cdb}$$

This identification should make clear the relation between gravity and geometry.

· <u>Principles of General Relativity</u>

1) Equivalence principle: in small enough regions of spacetime, the Laws of Physics reduce to those of Special Relativity. It is impossible to detect a gravitational field by means of local experiments

2) General Covariance: The Laws of Nature should have the same mathematical form in any reference frame → tensorial equations

3) Minimal coupling: the coupling to gravity is done by replacing $\partial \to \nabla$, $\eta_{ab} \to g_{ab}$

Example: perfect fluid in Special Relativity

$$T^{ab} = (\rho+p) V^a V^b - p\,\eta^{ab}$$

$$\partial_a T^{ab} = 0$$

Perfect fluid in a gravitational field:

$$T^{ab} = (\rho+p) V^a V^b - p\,g^{ab}$$

$$\nabla_a T^{ab} = 0$$

4) Correspondence principle: GR should reduce to SR in absence of gravity and should agree with the Newtonian theory of gravity in the case of weak gravitational fields and in the non-relativistic limit (small velocities compared to c)

· Einstein equations in vacuum

Equation for the Newtonian potential in absence of matter sources:

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^i \partial x^j} = 0$$

$\phi \leftrightarrow g_{ab}$

By analogy the equations for the gravitational field should involve a geometric object build from the

Riemann tensor ($\to$ 2nd derivatives of g_{ab}) and have the same number of components as g_{ab}.

One guesses:

$$R_{ab} = 0$$

$\to$ Einstein equations in vacuum

$\to$ Non-linear 2nd order PDEs for g_{ab}

Rmk: For Minkowski space in Cartesian coordinates,

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

$\to$ $g_{ab} = \mathrm{diag}(-1, 1, 1, 1) \Rightarrow \Gamma^a{}_{bc} = 0 \quad \partial_d \Gamma^a{}_{bc} = 0$

$\Rightarrow R_{ab} = 0$

Rmk: One can add a cosmological constant to the Einstein vacuum equations:

$$R_{ab} + \Lambda g_{ab} = 0$$

$\to$

- The full Einstein equations of GR

Matter in relativity is described by a $(0,2)$ tensor, T_{ab}, the stress-energy tensor that describes the distribution of matter/energy.

One would be tempted to generalise the Einstein equation as

$$R_{ab} = \kappa T_{ab}$$

for some coupling constant κ that determines the strength of gravity. However, this equation is inconsistent since mass-energy is conserved

$\nabla^a T_{ab} = 0$ but $\nabla^a R_{ab} \neq 0$

→ both sides of the Einstein equation must be covariantly conserved, therefore the only $(0,2)$ tensor constructed from the Ricci that has this property is the Einstein tensor:

$$G_{ab} = \kappa T_{ab}$$

since $\nabla^a G_{ab} = 0$ follows from the Bianchi id.

Note that $\nabla^a g_{ab} = 0$ (from metric compatibility) so we can generalise the equation above as

$$G_{ab} + \Lambda g_{ab} = \kappa T_{ab}$$

with $\kappa = 8\pi G$ to reproduce the Newtonian limit.

· The Schwarzschild solution

→ Unique spherically symmetric and static solution of the Einstein equations in vacuum:

$$R_{ab} = 0$$

· Staticity: there exists a Killing vector field such that far away from any sources reduces to ∂_t, which is the canonical timelike Killing vector field in Minkowski space. Furthermore, the metric must be invariant under $t \to -t$.

· Spherical symmetry: one can find coordinates such that the line element has an explicit round sphere: $d\Omega^2_{(2)} = d\theta^2 + \sin^2\theta \, d\phi^2$

The most general line element compatible with these symmetries is:

$$ds^2 = -e^{2A(r)} dt^2 + e^{2B(r)} dr^2 + r^2 e^{2C(r)} (d\theta^2 + \sin^2\theta \, d\phi^2)$$

We can further simplify this metric by using the freedom to redefine the radial coordinate:

$$\bar{r} = r e^{C(r)} \quad \rightarrow \quad d\bar{r} = e^{C(r)}\left(1 + r\frac{dC}{dr}\right) dr$$

$$\Rightarrow \quad ds^2 = -e^{2A(r)} dt^2 + (1+rC')^{-2} e^{2(B-C)} d\bar{r}^2 + \bar{r}^2 d\Omega^2_{(2)}$$

where now $r = r(\bar{r})$. Relabelling

$$\bar{r} \rightarrow r \, , \quad (1+rC')^{2(B(r)-C(r))} \rightarrow e^{2B(r)}$$

$$\Rightarrow \quad ds^2 = -e^{2A(r)} dt^2 + e^{2B(r)} dr^2 + r^2 d\Omega^2_{(2)} \qquad (*)$$

That's the most general static and spherically symmetric spacetime.

Now we solve the Einstein vacuum eqs for $(*)$

$$R_{ab} = 0$$

- Step 1 : compute the Christoffels

→ Use the Euler-Lagrange eqs:

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^a}\right) - \frac{\partial L}{\partial x^a} = 0 \quad \Leftrightarrow \quad \ddot{x}^a + \Gamma^a{}_{bc}\, \dot{x}^b \dot{x}^c = 0$$

$$L = -e^{2A(r)} \dot{t}^2 + e^{2B(r)} \dot{r}^2 + r^2\left(\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2\right)$$

$t)$: $\dfrac{\partial L}{\partial t} = 0$

$$\frac{\partial L}{\partial \dot t} = -2e^{2A}\dot t \; ; \; \frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot t}\right) = -2e^{2A}(\ddot t + 2A'\dot r \dot t)$$

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot t}\right) - \frac{\partial L}{\partial t} = -2e^{2A}(\ddot t + 2A'\dot r \dot t) = 0$$

$$\Rightarrow \quad \ddot t + 2A'\dot r \dot t = 0 \quad \Rightarrow \quad \Gamma^t{}_{tr} = \Gamma^t{}_{rt} = A'$$

$r)$ $\dfrac{\partial L}{\partial r} = 2\left[-e^{2A}A'\dot t^2 + e^{2B}B'\dot r^2 + r(\dot\theta^2 + \sin^2\theta\,\dot\phi^2)\right]$

$$\frac{\partial L}{\partial \dot r} = 2e^{2B}\dot r \; ; \; \frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot r}\right) = 2e^{2B}(\ddot r + 2B'\dot r^2)$$

$$0 = \frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot r}\right) - \frac{\partial L}{\partial r} = 2e^{2B}(\ddot r + 2B'\dot r^2) - 2\left[-e^{2A}A'\dot t^2 + e^{2B}B'\dot r^2 + r(\dot\theta^2 + \sin^2\theta\,\dot\phi^2)\right]$$

$$\Rightarrow \quad \ddot r + e^{2(A-B)}A'\dot t^2 + B'\dot r^2 - re^{-2B}(\dot\theta^2 + \sin^2\theta\,\dot\phi^2) = 0$$

$$\Rightarrow \quad \Gamma^r{}_{tt} = e^{2(A-B)}A' \; ; \; \Gamma^r{}_{rr} = B'$$

$$\Gamma^r{}_{\theta\theta} = -re^{-2B} \; ; \; \Gamma^r{}_{\phi\phi} = -r\sin^2\theta\, e^{-2B}$$

Similarly we compute:

$$\Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = \frac{1}{r} \; ; \; \Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} = \frac{1}{r} \; ; \; \Gamma^\phi{}_{\theta\phi} = \cot\theta$$

· Step 2 : compute the Ricci tensor

$$R_{ab} = \partial_c \Gamma^c{}_{ab} - \partial_a \partial_b \ln\sqrt{|g|} + \Gamma^c{}_{ab} \partial_c \ln\sqrt{|g|} - \Gamma^c{}_{ad} \Gamma^d{}_{bc}$$

$$\det g = - e^{2(A+B)} r^4 \sin^2\theta \rightarrow \sqrt{|g|} = e^{A+B} r^2 \sin\theta$$

$$\rightarrow \ln\sqrt{|g|} = A + B + 2 \ln r + \ln \sin\theta$$

· $$R_{tt} = \partial_c \Gamma^c{}_{tt} - \cancel{\partial_t^2 \ln\sqrt{|g|}}^{\,0} + \Gamma^c{}_{tt} \partial_c \ln\sqrt{|g|} - \Gamma^c{}_{td} \Gamma^d{}_{tc}$$

$$= \partial_r \Gamma^r{}_{tt} + \Gamma^r{}_{tt} \partial_r \ln\sqrt{|g|} - \Gamma^t{}_{tr} \Gamma^r{}_{tt} - \Gamma^r{}_{tt} \Gamma^t{}_{tr}$$

$$= \frac{d}{dr}\left(e^{2(A-B)} A' \right) + e^{2(A-B)} A' \left(A' + B' + \frac{2}{r} \right)$$
$$- 2 (A')^2 e^{2(A-B)}$$

$$= e^{2(A-B)} \left[A'' + A'\left(A' - B' + \frac{2}{r} \right) \right]$$

· $$R_{rr} = \partial_c \Gamma^c{}_{rr} - \partial_r^2 \ln\sqrt{|g|} + \Gamma^c{}_{rr} \partial_c \ln\sqrt{|g|} - \Gamma^c{}_{rd} \Gamma^d{}_{rc}$$

$$= \partial_r \Gamma^r{}_{rr} + \partial_r^2 \ln\sqrt{|g|} + \Gamma^r{}_{rr} \partial_r \ln\sqrt{|g|}$$
$$- \Gamma^t{}_{rt} \Gamma^t{}_{rt} - \Gamma^r{}_{rr} \Gamma^r{}_{rr} - \Gamma^\theta{}_{r\theta} \Gamma^\theta{}_{r\theta} - \Gamma^\phi{}_{r\phi} \Gamma^\phi{}_{r\phi}$$

$$= \cancel{B''} - \left(A'' + \cancel{B''} - \cancel{\frac{2}{r^2}} \right) + B'\left(A' + \cancel{B'} + \frac{2}{r} \right)$$
$$- (A')^2 - \cancel{(B')^2} - \cancel{\frac{2}{r^2}}$$

$$= - A'' - (A')^2 + A'B' + \frac{2}{r} B'$$

Similarly we compute

$$R_{\theta\theta} = e^{-2B}[r(B'-A')-1]+1$$

$$R_{\phi\phi} = \sin^2\theta \, R_{\theta\theta} \quad \text{by spherical symmetry.}$$

- Step 3 : Solve the Einstein equations $R_{ab}=0$

$$R_{tt} = e^{2(A-B)}\left(A'' + A'^2 - A'B' + \frac{2}{r}A'\right)$$

$$R_{rr} = -A'' - A'^2 + A'B' + \frac{2}{r}B'$$

$$R_{\theta\theta} = e^{-2B}[r(B'-A')-1]+1$$

Since all components of Ricci have to vanish independently we can consider linear combinations:

$$0 = e^{2(B-A)}R_{tt} + R_{rr} = \frac{2}{r}(A'+B')$$

$$\Rightarrow \quad A(r) = -B(r) + c \quad , \quad c = \text{const}$$

We can set $c=0$ by rescaling $t \to e^{-c}t$ so

$$A(r) = -B(r)$$

$$0 = R_{\theta\theta} = e^{2A(r)}(-2rA'-1)+1$$

$$\Rightarrow \quad e^{2A(r)}(2rA'+1) = 1$$

$$\Rightarrow \quad \frac{d}{dr}\left(r\,e^{2A(r)}\right) = 1 \quad \Rightarrow \quad e^{2A(r)} = 1 - \frac{R_s}{r}$$

R_s : constant

With these results the line element becomes:

$$ds^2 = -\left(1 - \frac{R_s}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{R_s}{r}} + r^2 d\Omega^2_{(2)}$$

We can fix R_s by requiring that in the weak field regime, $r \gg R_s$, we recover the previous results:

$$g_{tt} = -\left(1 - \frac{R_s}{r}\right) = -(1 + 2\phi) = -\left(1 - \frac{2GM}{r}\right)$$

$\Rightarrow R_s = 2GM$

We can finally write down the final form of a static, spherically symmetric vacuum spacetime:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2GM}{r}} + r^2 d\Omega^2_{(2)}$$

$\rightarrow$ Schwarzschild metric

M: mass of the spacetime. For $M \to 0$ we recover Minkowski space

- This spacetime describes the exterior of a star. It also describes a black hole

- <u>Birkhoff's Thm</u>: a spherically symmetric solution of the Einstein eqs in vacuum is necessarily static.