Given a posterior and loss function the Bayesian estimator $T(\underline{y})$ is the $T(\underline{y})$ which which minimizes the expected loss with respect to the posterior, where the expected posterior loss (EPL) is defined to be

$$E[L(T(\underline{y}), \underline{\theta})]$$

where E is taken with respect to the posterior on θ. (data is $\underline{y}$).

$$E[L(T(\underline{y}), \underline{\theta})] = \int L(T(\underline{y}), \underline{\theta}) f(\underline{\theta}|\underline{y}) d\underline{\theta}$$

We will be taking $p=1$, θ only parameter.

We want to find $T(\underline{y})$ which minimizes EPL.

There are three loss functions we will consider.

Let $\phi(\theta) = \theta$

1. Quadratic Loss:

$$L(T(\underline{y}), \theta) = (T(\underline{y}) - \theta)^2$$

2. Absolute Error Loss

$$L(T(\underline{y}), \theta) = |T(\underline{y}) - \theta|$$

3. All-or-nothing Loss

$$L(T(\underline{y}), \theta) = \begin{cases} 0 & \text{if } T(\underline{y}) = \theta \\ 1 & \text{if } T(\underline{y}) \neq \theta \end{cases}$$

We will find Bayesian estimators for these three loss functions

1. Quadratic.

$$EPL = \int (T(\underline{y}) - \theta)^2 f(\theta | \underline{y}) \, d\theta$$

$$\frac{d\,EPL}{d\,T(\underline{y})} = \int 2(T(\underline{y}) - \theta) f(\theta | \underline{y}) \, d\theta = 0$$

$$T(\underline{y}) \int f(\theta | \underline{y}) \, d\theta = \int \theta f(\theta | \underline{y}) \, d\theta$$

$$T(\underline{y}) = \int \theta f(\theta | \underline{y}) \, d\theta$$

$$:= E(\theta | \underline{y})$$

In an example, we found the posterior for λ

posterior $\sim$ Gamma $(26, 10)$

Under quadratic loss, the Bayes estimate is $\frac{26}{10} = 2.6$

if ~~one of the parameters~~ is known.

2. Absolute Error Loss

$$EPL = \int |T(\underline{y}) - \theta| \, f(\theta | \underline{y}) \, d\theta$$

$$= \int_{-\infty}^{T(\underline{y})} (T(\underline{y}) - \theta) \, f(\theta | \underline{y}) \, d\theta$$

$$+ \int_{T(\underline{y})}^{\infty} (\theta - T(\underline{y})) \, f(\theta | \underline{y}) \, d\theta$$

$$\frac{\partial EPL}{\partial T(\underline{y})} = \int_{-\infty}^{T(\underline{y})} 1 \, f(\theta | \underline{y}) \, d\theta$$

$$+ \int_{T(\underline{y})}^{\infty} (-1) \, f(\theta | \underline{y}) \, d\theta = 0$$

$$\Rightarrow \int_{-\infty}^{T(\underline{y})} f(\theta | \underline{y}) \, d\theta = \int_{T(\underline{y})}^{\infty} f(\theta | \underline{y}) \, d\theta$$

$\Rightarrow$ $T(\underline{y})$ is the median of the posterior.

If the posterior for λ is Gamma(26,10)
the Bayesian estimator 2.5667
using qgamma in R.

3. All-or-nothing loss

We use an ~~indirect~~ indirect approach.

Consider the loss function

$$L_\epsilon \cancel{(T(\underline{y}), \theta) = \begin{cases} 0 \text{ if } |\theta - T(\underline{y})| < \epsilon \end{cases}}$$

$$L_\epsilon(T(\underline{y}), \theta) = \begin{cases} 0 & \text{if } |\theta - T(\underline{y})| < \epsilon \\ 1 & \text{else} \end{cases}$$

$$EPL(L_\epsilon) = 1 - \int_{T(\underline{y})-\epsilon}^{T(\underline{y})+\epsilon} f(\theta|\underline{y})\, d\theta$$

because $1 = \int f(\theta|\underline{y})\, d\theta$

so $EPL(L_\epsilon) \cong 1 - 2\epsilon\, f(T(\underline{y})|\underline{y})$

This is minimized when $f(T(\underline{y})|\underline{y})$ is maximized, i.e. at the mode of $f(\theta|\underline{y})$.

3

Example

Posterior is Gamma (26, 10)

$$f(\theta | \underline{y}) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \theta^{\alpha - 1} e^{-\lambda\theta}$$

where $\alpha = 26$

$\theta = 10$

We want to maximize $f(\theta | \underline{y})$

instead we maximize $\ln f(\theta | \underline{y})$

$$\ln f(\theta | \underline{y}) = (\alpha - 1)\ln(\theta) - \lambda\theta + \ln\left(\frac{\lambda^{\alpha}}{\Gamma(\alpha)}\right)$$

$$\frac{d}{d\theta} \ln f(\theta | \underline{y}) = \frac{\alpha - 1}{\theta} - \lambda = 0$$

$$\theta = \frac{\alpha - 1}{\lambda}$$

is the mode

$$\theta = \frac{25}{10} = 2.5$$

Credibility Theory

Credibility is framework in the posterior can often be expressed, the framework has a nice interpretation.

First, we need to review conditional expectation.

$$E[X \mid Y=y] = \sum x \, P(X=x \mid Y=y) \quad \text{if } X, Y \text{ discrete}$$

$$= \int_{-\infty}^{\infty} x \, \frac{f_{X,Y}(x,y)}{f_Y(y)} \, dx \quad \text{if } X, Y \text{ continuous}$$

We let $E[X \mid Y]$ equal $E[X \mid Y=y]$ when $Y=y$.

It is a function of Y.