MTH5113 (2023/24): Problem Sheet 7

All coursework should be submitted individually.

- Problems marked "[**Marked**]" should be submitted and will be marked.

Please submit the completed problem on QMPlus:

- At the portal for **Coursework Submission 2**.

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(1) *(Warm-up)* For each of the parametric surfaces σ given below and every pair of parameters (u, v) in the domain of σ, compute the following:

(i) $\partial_1\sigma(u, v)$ and $\partial_2\sigma(u, v)$.

(ii) $\partial_1\sigma(u, v) \times \partial_2\sigma(u, v)$.

(iii) $|\partial_1\sigma(u, v) \times \partial_2\sigma(u, v)|$.

(a) *Sphere:* $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$, where $\sigma(u, v) = (\cos u \sin v,\ \sin u \sin v,\ \cos v)$.

(b) *Torus:* $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$, where $\sigma(u, v) = ((2 + \cos u) \cos v,\ (2 + \cos u) \sin v,\ \sin u)$.

(2) *(Warm-up)* Determine whether the following parametric surfaces are regular:

(a) *Paraboloid:*

$$\sigma : \mathbb{R}^2 \to \mathbb{R}^3, \qquad \sigma(u, v) = (u,\ v,\ u^2 + v^2).$$

(b) *(Polar)* xy*-plane:*

$$\mathbf{P} : \mathbb{R}^2 \to \mathbb{R}^3, \qquad \mathbf{P}(u, v) = (u \cos v,\ u \sin v,\ 0).$$

(c) *One-sheeted hyperboloid:*

$$\mathbf{H} : \mathbb{R}^2 \to \mathbb{R}^3, \qquad \mathbf{H}(u, v) = (\cos u \cosh v,\ \sin u \cosh v,\ \sinh v).$$

(3) *(Parametrise me!)* For each surface S and point $\mathbf{p} \in S$ below, give a parametrisation σ of S such that $\mathbf{p}$ lies in the image of σ.

(a) *Plane:*

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid y = z\}, \qquad \mathbf{p} = (1, -4, -4).$$

(b) *Ellipsoid:*

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + 4y^2 + 4z^2 = 4\}, \qquad \mathbf{p} = (2, 0, 0).$$

(c) *Gabriel's Horn:*

$$S = \left\{(x, y, z) \in \mathbb{R}^3 \mid x > 0,\ y^2 + z^2 = \frac{1}{x^2}\right\}, \qquad \mathbf{p} = \left(1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

(4) [**Marked**] Consider the following set:

$$S = \left\{(x, y, z) \in \mathbb{R}^3 \mid \left(3 + y^2\right)\left(x^2 + z^2\right) = 1\right\}.$$

(a) Show that S is a surface.

(b) Sketch S.

(c) Give a parametrisation of S such that $\left(-\frac{1}{2\sqrt{2}}, 1, \frac{1}{2\sqrt{2}}\right)$ lies in the image of S.

(d) Compute the tangent plane to S at $\left(-\frac{1}{2\sqrt{2}}, 1, \frac{1}{2\sqrt{2}}\right)$.

(5) [**Tutorial**] Consider the *two-sheeted hyperboloid*:

$$\mathcal{H} = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = -1\}.$$

(a) Show that $\mathcal{H}$ is a surface.

(b) Give a sketch of $\mathcal{H}$.

(c) Give a parametrisation of $\mathcal{H}$ that passes through the point $(1, -1, \sqrt{3})$.

(d) Compute the tangent plane to $\mathcal{H}$ at the point $(1, -1, \sqrt{3})$.

(6) *(Let's be self-sufficient)* For each of the following surfaces S and points $\mathbf{p} \in S$:

(i) Show that S is a surface.

(ii) Compute the tangent plane to S at $\mathbf{p}$.

(Unlike in Questions (4) and (5), you are not given a parametrisation of S. You will have to find your own in order to compute the tangent plane.)

(a) *Hyperbolic paraboloid:*

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x = yz\}, \qquad \mathbf{p} = (-6, 2, -3).$$

(b) *Cylinder:*

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + z^2 = 9\}, \qquad \mathbf{p} = \left(-\frac{3}{\sqrt{2}}, 7, \frac{3}{\sqrt{2}}\right).$$

(7) *(Surfaces of revolution)* Let $f : (a, b) \to \mathbb{R}$ be a smooth function satisfying $f(x) > 0$ for every $x \in (a, b)$. From f, we can then define the set

$$\mathcal{R} = \{(x, y, z) \in \mathbb{R}^3 \mid a < x < b,\ y^2 + z^2 = [f(x)]^2\}.$$

In particular, $\mathcal{R}$ is the *surface of revolution* obtained by taking the graph of f (in the xy-plane) and rotating it (in 3-dimensional space) around the x-axis.

(a) Show that $\mathcal{R}$ is indeed a surface.

(b) Give a parametrisation of $\mathcal{R}$ whose image is all of $\mathcal{R}$.

(c) Compute the tangent plane to $\mathcal{R}$ at the point $(x, 0, f(x))$, for any $x \in (a, b)$.

(8) *(Fun with stereographic projections)* Consider the parametric surface

$$\sigma : \mathbb{R}^2 \to \mathbb{R}^3, \qquad \sigma(u, v) = \mathbf{p},$$

where $\mathbf{p}$ is the (unique) point of $\mathbb{S}^2 \setminus \{(0, 0, 1)\}$ that lies on the line through the points $(u, v, 0)$ and $(0, 0, 1)$. (The function σ is called the *inverse stereographic projection.*)

(a) Show that σ can be described by the formula

$$\sigma(u, v) = \left(\frac{2u}{1 + u^2 + v^2}, \frac{2v}{1 + u^2 + v^2}, \frac{-1 + u^2 + v^2}{1 + u^2 + v^2}\right), \qquad (u, v) \in \mathbb{R}^2.$$

(b) Show that σ is both injective and regular.

(c) Show that the image of σ is precisely $\mathbb{S}^2 \setminus \{(0, 0, 1)\}$.

(d) Use your knowledge of σ to construct the sphere $\mathbb{S}^2$ using only two regular and injective parametric surfaces.