Week 8 Ruin Theory

Def: Successive time periods

Surplus

Notations: $N(t)$: the number of claims ~~genera~~ in the time interval $[0, t]$

$N \sim Poisson(\lambda)$

$N(t) \sim Poisson(\lambda t)$

X_i: the amount of the i^{th} claim, $i = 1, 2, 3, \cdots$

$S(t)$: the aggregate claims in the time interval $[0, t]$ for all $t \geq 0$

$\{N(t)\}_{t \geq 0}$, $\{S(t)\}_{t \geq 0}$ are stochastic processes.

$$S(t) = \sum_{i=1}^{N(t)} X_i \quad \xleftarrow{t} \quad S = \sum_{i=1}^{N} X_i$$

$S(t)$ is 0 if $N(t)$ is 0

C = the rate of premium income per unit time. $C > 0$

The Surplus process ← Insurers context

Surplus at time t : $U(t)$

Initial surplus : u

Formula for $U(t)$: $$U(t) = \underset{\substack{\uparrow \\ \text{initial} \\ \text{surplus}}}{u} + \underset{\substack{\uparrow \\ \text{Premium} \\ \text{incom}}}{ct} - \underset{\substack{\uparrow \\ \text{aggregate} \\ \text{claim} \\ \text{payment}}}{S(t)}$$

Surplus $U(t) < 0$ $\Rightarrow$ ruin ~~occu~~ occurred

insolvency

need more capital

Continuos time : prob of ruin

Def :

$$\psi(u) = P[U(t) < 0, \text{ for some } t, \ 0 < t < \infty] \leftarrow$$ Ruin in infinite time
Ultimate ruin
$[0, \infty)$

$$\psi(u, t) = P[U(\tau) < 0, \text{ for some } \tau, \ 0 < \tau < t] \leftarrow$$ ruin within time t
$[0, t)$
Ruin in finite time

Relationship:

$0 < t_1 \leq t_2 < \infty$, $0 \leq u_1 \leq u_2$

$U(t) = u + ct - S(t)$

1. $\psi(u_2, t) \leq \psi(u_1, t)$ $[0, t)$ $P[U(t) < 0]$

more initial surplus

2. $\psi(u_2) \leq \psi(u_1)$ $[0, \infty)$

3. $\psi(u, t_1) \leq \psi(u, t_2) \leq \psi(u)_\infty$ longer time

4. $\lim_{t \to \infty} \psi(u, t) = \psi(u)$

5. $\lim_{u \to \infty} \psi(u, t) = 0$

Ruin probability in discrete time

check for ruin at discrete time

$\psi_h(u) = P[\,u(t) < 0,$ for some time t, $t = h, 2h, 3h, \cdots]$

$\psi_h(u, t) = P[\,u(\tau) < 0,$ for some τ, $\tau = h, 2h, 3h, \cdots, t-h, t\,]$

Relationship:

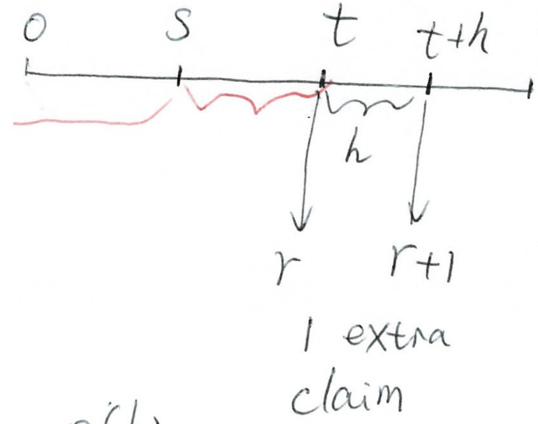
1. $\psi_h(u_2, t) \le \psi_h(u_1, t)$
2. $\psi_h(u_2) \le \psi_h(u_1)$
3. $\psi_h(u, t_1) \le \psi_h(u, t_2) \le \psi_h(u)$
4. $\lim_{t \to \infty} \psi_h(u, t) = \psi_h(u)$
5. $\psi_h(u,t) \le \psi(u,t)$ The more often we check, The more likely we are to find it

The Poisson process $N(t)$

Claim number process $\{N(t)\}_{t\geq 0}$: a Poisson process

Parameter λ

Statisfy the following conditions.



1. $N(0)=0$, $N(s) \leq N(t)$ when $s<t$

2. $P(N(t+h)=r \mid N(t)=r) = 1-\lambda h + o(h)$

   $P(N(t+h)=r+1 \mid N(t)=r) = \lambda h + o(h)$

   $P(N(t+h)>r+1 \mid N(t)=r) = o(h)$ $\lim_{h\to 0} \frac{o(h)}{h} = 0$

3. when $s<t$

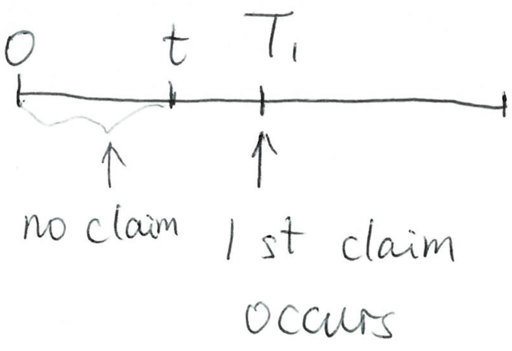
   The number of claims ~~N(t)~~ in the interval $(s,t]$ is independent of the number of claims up to time s

   ↳ independence of the past/history

Q1: Time to the first claim / waiting time $T_1 \sim Exp(\lambda)$

$N(t) \sim Poisson(\lambda t)$

T_1 : time of the first claim



If $T_1 > t$: no claim between $(0, t)$

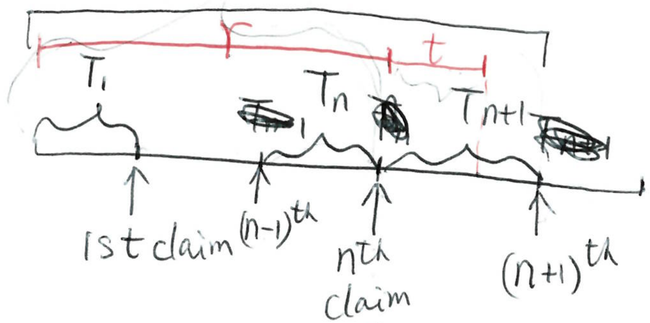
$P(T_1 > t) = P(N(t) = 0) = e^{-\lambda t}$

$N(t) \sim Poisson(\lambda t)$

$P(T_1 \leq t) = \underbrace{1 - e^{-\lambda t}}$ $\longrightarrow$ CDF of Exponential (λ)

$T_1 \sim Exp(\lambda)$

Q2: Time between claims



For $i = 1, 2, 3, \ldots$

r.Vs T_i : time between the $(i-1)^{th}$ and i^{th} claims. $\sum_{i=1}^{n} T_i$

$$P\left(\sum_{i=1}^{n+1} T_i > t+r \;\middle|\; \sum_{i=1}^{n} T_i = r\right)$$

$$= P(N(t+r)=n \mid N(r)=n)$$ no additional claims during $(r, r+t)$

$$= P(N(t+r)-N(r)=0 \mid N(r)=n)$$

Fact: claim numbers in different time periods are $\boxed{\text{independent}}$

$$= P(N(t+r)-N(r)=0)$$

Fact: the num of claims ~~at~~ in a time interval of length t does not depend on when it starts

$$= P(N(t)=0)$$ does not depend on r, only depends on $(t+r)-r=t$

$$= e^{-\lambda t}$$

$\boxed{N(t) \sim \text{Poisson}(\lambda t)}$ $t\uparrow,\ N(t)\uparrow$

time between claims (inter-event times) $\sim \exp(\lambda)$

E.g. Slide 17

Unit: day

Poisson, $\lambda = 5$ $\quad N(t) \sim \text{Poisson}(5t)$, $N(1) \sim \text{Poisson}(5)$

(i) $\text{Prob}(N(1) < 2) = P(N(1) = 0) + P(N(1) = 1)$

$= e^{-5} + 5e^{-5} = 4\%$

(ii) Another claim reported during the next hour $t = \frac{1}{24}$

$P(T < \frac{1}{24}) = 1 - e^{-\lambda t}$

$= 1 - e^{-5 \times \frac{1}{24}}$

$= 18.8\%$

$P(N(t) = 0) = e^{-\lambda t}$

no claim between t

$P(N(t) \geq 1) = 1 - e^{-\lambda t}$

Slide 19 - 24 read

Similar to Week 3 & 4

~~N~~ $N \to N(t)$

$\lambda \to \lambda t$

Premium security loadings

$c = (1+\theta)\lambda m_1$ $\qquad$ $c = \cancel{\theta}\, \lambda m_1$

Expectation/mean

premium loading factor

Assumption: $\theta > 0$

Lundberg's inequality

$$\psi(u) \le e^{-Ru}$$

R: adjustment coefficient

Visulisation of $\psi(u) \le e^{-Ru}$: Slide 27

① $\psi(u) \le e^{-Ru}$

② $\psi(u) \to e^{-Ru}$ close for large u

Interpretation

① e^{-Ru} as an approximation to $\psi(u)$. $\psi(u) \simeq e^{-Ru}$

② R: risk measure, R is an inverse risk measure $R\uparrow$, $\psi(u)\downarrow$, Risk$\downarrow$

③ $R\uparrow$, upper bound for $\psi(u)\downarrow$, $\psi(u)\downarrow$ when $R\uparrow$

$\psi(u) \approx e^{-Ru}$

$\cancel{R(\theta)}$ $R = f(\theta)$ $\quad \theta \uparrow.$ $\quad R?$

$\theta \uparrow.$ $\quad c = (1+\theta)\lambda m_1 \uparrow$, $\quad R$ inverse of risk $\uparrow$ $\Rightarrow \theta\uparrow, R\uparrow$

risk $\downarrow$

The adjustment coefficient R:

$\psi(u) \leq e^{-Ru}$

$$\boxed{\lambda M_X(R) - \lambda - cR = 0}$$

R is defined to be the unique positive root of this equation

or $\cancel{R}$ $\lambda M_X(R) = \lambda + cR$

R depends on: λ, X_i distribution, c premium income

$c = (1+\theta)\lambda m_1$

$\lambda M_X(R) - \lambda - (1+\theta)\lambda m_1 R = 0$

$$\boxed{M_X(R) = 1 + (1+\theta) m_1 R}$$

R depends on: θ, X_i distribution

Example 1 Slide 33 Exam-style question

$\lambda = 15$, $X \sim \exp\left(\frac{1}{500}\right)$, $\theta = 30\%$, $U = 1000$

Calculate R.

upper bound $\psi(u)$.

A: $\boxed{M_X(R) = 1 + (1+\theta) m_1 R}$

$X \sim \text{Exp}\left(\frac{1}{500}\right)$, $M_X(R) = \frac{1}{1-500R}$ Week 1 Slide 14-15

$\theta = 0.3$, $m_1 = E(X) = 500$

$$\frac{1}{1-500R} = 1 + 1.3 \times 500R$$

$R = 0.000462$

From Lundberg's inequality $\psi(u) \le e^{-Ru}$, we get $\psi(u) \le e^{-0.000462 \times 1000} = 0.630$

Note: λ is not used

Example 2. $X_i \sim Exp(\alpha)$ Exam - Style question

$$F(x) = 1 - e^{-\alpha x} \qquad M_X(R) = \frac{\alpha}{\alpha - R}$$

$$\lambda M_X(R) = \lambda + CR$$

$$\lambda \frac{\alpha}{\alpha - R} = \lambda + CR$$

$$\begin{cases} R = \alpha - \dfrac{\lambda}{C} \\ C = (1+\theta)\lambda m_1 = (1+\theta)\lambda \cdot \dfrac{1}{\alpha} \end{cases}$$

$$R = \frac{\alpha\theta}{1+\theta}$$

Example 3. Exam - style question
Slide 37.

$$X = \begin{cases} 10{,}000 & \text{prob } 0.9 \\ 25{,}000 & \text{prob } 0.1 \end{cases}$$

$\theta = 20\%$

Q: $0.00002599 < R < 0.00002601$

A:

$$\boxed{1 + (1+\theta)\, m_1 R = M_X(R)}$$

$m_1 = E(X) = \sum x P(X=x) = 0.9 \times 10{,}000 + 0.1 \times 25{,}000 = 11{,}500$

$M_X(R) = E(e^{RX}) = \sum e^{Rx} P(X=x) = 0.9 e^{10{,}000R} + 0.1 e^{25{,}000R}$

$\theta = 0.2$

$$1 + 1.2 \times 11{,}500 R = 0.9 e^{10{,}000R} + 0.1 e^{25{,}000R}$$

$$1 + 13800R = 0.9 e^{10{,}000R} + 0.1 e^{25000R}$$

$L(R) = \text{LHS} - \text{RHS}$

$R = 0.00002599$, $L(R) = 1 + 13{,}800R - 0.9 e^{10{,}000R} - 0.1 e^{25000R}$

$= 0.000035 > 0$

$R = 0.00002601$, $L(R) = 1 + 13{,}800R - 0.9 e^{10{,}000R} - 0.1 e^{25000R}$

$= -0.000018 < 0$

$L(R)$ is a $\boxed{\text{continuous}}$ function.

$L(R)$ must be 0 at some point between the two values.

$0.00002599 < R < 0.00002601$. ▨

An upper bound of R

$$\lambda + cR = \lambda M_X(R)$$

$$= \lambda \int_0^\infty e^{RX} f(x)\,dx$$

$$> \lambda \int_0^\infty \boxed{\left(1 + RX + \frac{1}{2}R^2X^2\right)} f(x)\,dx$$

$$= \lambda \int_0^\infty f(x)\,dx + \lambda \int_0^\infty RX f(x)\,dx + \frac{\lambda}{2} \int_0^\infty R^2X^2 f(x)\,dx$$

$$= \lambda \times 1 + \lambda R m_1 + \frac{\lambda}{2} R^2 m_2$$

$$\lambda + cR > \lambda\left(1 + Rm_1 + \frac{1}{2}R^2 m_2\right)$$

$$(c - \lambda m_1)R > \frac{1}{2}\lambda R^2 m_2$$

$$R < \frac{2(c - \lambda m_1)}{\lambda m_2} \qquad c = (1+\theta)\lambda m_1$$

$$\boxed{R < \frac{2\theta m_1}{m_2}}$$

A lower bound for R

Extra assumption: individual claim amount X_i has an upper limit, M

$$X_i \leq M$$

$$e^{RX} \leq \frac{X}{M} e^{RM} + 1 - \frac{X}{M} \quad (*), \quad 0 \leq X \leq M$$

$$\text{RHS} = \frac{X}{M} \sum_{j=0}^{\infty} \frac{(RM)^j}{j!} + 1 - \frac{X}{M}$$

$$= \frac{X}{M} \left[\sum_{j=0}^{\infty} \frac{(RM)^j}{j} - 1 \right] + 1$$

Start with 1 when $j=0$

$$= \frac{X}{M} \cdot \sum_{j=1}^{\infty} \frac{(RM)^j}{j!} + 1$$

$$= 1 + \sum_{j=1}^{\infty} \frac{R^j M^{j-1} X}{j!} \qquad X \leq M$$

$$\geq 1 + \sum_{j=1}^{\infty} \frac{(RX)^j}{j!} = \sum_{j=0}^{\infty} \frac{(RX)^j}{j!} = e^{RX} = \text{LHS}$$

Use Inequation $(*)$

$$\lambda + CR = \lambda M_X(R) = \lambda \int_0^M e^{Rx} f(x)\, dx$$

$$\overset{(*)}{\leq} \lambda \int_0^M \left(\frac{x}{M} e^{RM} + 1 - \frac{x}{M} \right) f(x)\, dx$$

$$= \lambda \int_0^M \frac{e^{RM}}{M} x\, f(x)\, dx + \lambda \int_0^M f(x)\, dx$$

$$- \frac{\lambda}{M} \int_0^M x\, f(x)\, dx$$

$$\cancel{\lambda} + CR \leq \frac{\lambda e^{RM}}{M} m_1 + \cancel{\lambda} - \frac{\lambda}{M} m_1$$

$$\frac{c}{\lambda m_1} \leq \frac{1}{RM} \left(e^{RM} - 1 \right) = 1 + \frac{RM}{2} + \frac{(RM)^2}{3!} + \cdots$$

$$< 1 + \frac{RM}{1!} + \frac{(RM)^2}{2!} + \cdots$$

$$= e^{RM}$$

$$\frac{c}{\lambda m_1} < e^{RM} \Rightarrow R > \frac{1}{M} \ln\left(\frac{c}{\lambda m_1} \right)$$