1

Let the coordinates be:

$$x^a = (u, r, \theta, \varphi)$$

Then, we have:

$$g_{uu} = \frac{V}{r} e^{2\beta} - g^2 r^2 e^{2\alpha}$$

$$g_{ur} = e^{2\beta}$$

$$g_{u\theta} = g\, r^2 e^{2\alpha}$$

$$g_{\theta\theta} = - r^2 e^{2\alpha}$$

$$g_{\varphi\varphi} = - r^2 e^{-2\alpha} \sin^2\theta$$

The remaining components vanish. In matrix form:

$$g_{ab} = \begin{pmatrix} \frac{V}{r} e^{2\beta} - g^2 r^2 e^{2\alpha} & e^{2\beta} & g\, r^2 e^{2\alpha} & 0 \\ e^{2\beta} & 0 & 0 & 0 \\ g\, r^2 e^{2\alpha} & 0 & -r^2 e^{2\alpha} & 0 \\ 0 & 0 & 0 & -r^2 e^{-2\alpha} \sin^2\theta \end{pmatrix}$$

(2) Line element of $\mathbb{R}^3$ in spherical coordinates:

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2$$

The metric g_{ab} and the inverse metric g^{ab} are:

$$g_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix} , \quad g^{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1/(r^2 \sin^2\theta) \end{pmatrix}$$

a) $X^a = (1, r, r^2)$

$\rightarrow X_a = g_{ab} X^b = g_{ar} + r\, g_{a\theta} + r^2 g_{a\varphi}$

$$\Rightarrow \left.\begin{aligned} X_r &= g_{rr} = 1 \\ X_\theta &= r\, g_{\theta\theta} = r^3 \\ X_\varphi &= r^2 g_{\varphi\varphi} = r^4 \sin^2\theta \end{aligned}\right\} X_a = (1, r^3, r^4 \sin^2\theta)$$

b) $Y_a = (0, -r^2, r^2 \cos^2\theta)$

$Y^a = g^{ab} Y_b$

$Y^r = g^{rr} Y_r = 0$

$Y^\theta = g^{\theta\theta} Y_\theta = \frac{1}{r^2}(-r^2) = -1$

$Y^\varphi = g^{\varphi\varphi} Y_\varphi = \frac{1}{r^2 \sin^2\theta}(r^2 \cos^2\theta) = \cot^2\theta$

(3) T_{ab} is conserved: $\nabla^a T_{ab} = 0$ and $T_{ab} = T_{(ab)}$

X^a satisfies: $\nabla_{(a} X_{b)} = 0$

$V_a = T_{ab} X^b$

Then,

$$\nabla^a V_a = \nabla^a (T_{ab} X^b) =$$
$$= (\nabla^a T_{ab}) X^b + T_{ab} \nabla^a X^b$$
$$= 0 + T_{ab} \nabla^{(a} X^{b)}$$
$$= 0$$

To go from the first line to the second we have used the Leibniz rule, and in the third line we have used that T_{ab} is conserved and $T_{ab} \nabla^a X^b = T_{(ab)} \nabla^{(a} X^{b)}$ because T_{ab} is symmetric. In the fourth line we use that $\nabla_{(a} X_{b)} = 0$ is a tensor equation and we can raise the indices with g^{ab}.

④ $S_b{}^c$ a $(1,1)$ tensor

a) It follows from the general formula given in the lectures:

$$\nabla_a S_b{}^c = \partial_a S_b{}^c + \Gamma^c{}_{da} S_b{}^d - \Gamma^d{}_{ba} S_d{}^c$$

b) Using the previous formula,

$$\nabla_a \delta_b{}^c = \partial_a \delta_b{}^c + \Gamma^c{}_{da} \delta_b{}^d - \Gamma^d{}_{ba} \delta_d{}^c$$

$$= \Gamma^c{}_{ba} - \Gamma^c{}_{ba} = 0$$

where we have used that $\partial_a \delta_b{}^c = 0$ since the components of $\delta_a{}^b$ are constants.

(Notice that for a metric compatible connection,

$$\nabla_a \delta_b{}^c = \nabla_a (g_{bd} \, g^{dc}) = 0$$

since $\nabla_a g_{bc} = 0$ and $\nabla_a g^{bc} = 0$)

c) $$\delta_a{}^a = \sum_{a=0}^{3} \delta_a{}^a = \sum_{a=0}^{3} 1 = 4$$

(5) Starting from, $g_{ab} g^{bc} = \delta_a{}^c$ and applying ∇_d, we get

$$\nabla_d (g_{ab} g^{bc}) = (\cancel{\nabla_d} g_{ab})^{0} g^{bc} + g_{ab} \nabla_d g^{bc} =$$
$$= \nabla_d \delta_a{}^c = 0$$

$$\Rightarrow \quad g_{ab} \nabla_d g^{bc} = 0$$

Multiplying by g^{ea} we get

$$g^{ae} g_{ab} \nabla_d g^{bc} = \delta^e{}_b \nabla_d g^{bc} = \nabla_d g^{ec} = 0$$

which is the desired result

(6) Given the line element,

$$ds^2 = e^y dx^2 + e^x dy^2$$

we have:

a) $$g_{ab} = \begin{pmatrix} e^y & 0 \\ 0 & e^x \end{pmatrix} , \quad g^{ab} = \begin{pmatrix} e^{-y} & 0 \\ 0 & e^{-x} \end{pmatrix}$$

b) Recall the general formula:

$$\Gamma^a{}_{bc} = \frac{1}{2} g^{ad} (\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc})$$

Using the identification $(x^1, x^2) = (x, y)$, we have

$$\Gamma^1_{11} = \frac{1}{2} g^{1d} (\partial_1 g_{1d} + \partial_1 g_{1d} - \partial_d g_{11})$$
$$= \frac{1}{2} g^{11} \partial_1 g_{11} = 0$$

$$\Gamma^1_{12} = \frac{1}{2} g^{1d} (\partial_1 g_{2d} + \partial_2 g_{1d} - \partial_d g_{12})$$
$$= \frac{1}{2} g^{11} \partial_2 g_{11} = \frac{1}{2} e^{-y} e^{y} = \frac{1}{2}$$

$$\Gamma^2_{11} = \frac{1}{2} g^{2d} (2 \partial_1 g_{1d} - \partial_d g_{11})$$
$$= -\frac{1}{2} g^{22} \partial_2 g_{11} = -\frac{1}{2} e^{y-x}$$

(7) Since X^a is the tangent vector to a geodesic, it satisfies

$$X^b \nabla_b X^a = 0$$

(wlog here we assume that the geodesic is affinely parametrised). Then, we have

$$X^a \nabla_a(|X|^2) = X^a \nabla_a (g_{bc} X^b X^c)$$
$$= X^a \Big[(\nabla_a g_{bc}) X^b X^c$$
$$+ g_{bc} (\nabla_a X^b) X^c + g_{bc} X^b (\nabla_a X^c) \Big]$$

$$= 2 X_b X^a \nabla_a X^b$$
$$= 0$$

To go from the 2nd to the 3rd line we have used that

$\nabla_a g_{bc} = 0$ (metric compatible connection)

$$g_{bc}\left((\nabla_a X^b) X^c + X^b \nabla_a X^c\right) = 2 g_{bc} X^b \nabla_a X^c$$
$$= 2 X_b \nabla_a X^b$$

since g_{bc} is symmetric

(8) V^a a Killing vector: $\nabla_{(a} V_{b)} = 0$

X^a tangent vector to a geodesic: $X^a \nabla_a X^b = 0$

E a scalar: $E \equiv V_a X^a$

Then,

$$X^a \nabla_a E = X^a \nabla_a (V_b X^b)$$
$$= X^a X^b \nabla_a V_b + V_b \underbrace{\cancel{X^a \nabla_a X^b}}_{0}$$

X^a tangent to geodesic

$$= X^a X^b \nabla_{(a} V_{b)}$$
$$= 0$$

In going from the 2nd to the 3rd line we have used that X^a is tangent to a geodesic and hence it satisfies $X^a \nabla_a X^b = 0$.

In addition, note that $X^a X^b$ is symmetric under $a \leftrightarrow b$. Hence, in the term $X^a X^b \nabla_a V_b$ only the symmetric part of $\nabla_a V_b$ contributes:

$$X^a X^b \nabla_a V_b = X^a X^b \nabla_{(a} V_{b)}$$

In the last line we have used Killing's equation:

$$\nabla_{(a} V_{b)} = 0$$

9

$$ds^2 = -e^{2Ar} dt^2 + dr^2 \quad , \quad A: \text{constant}$$

$$g_{ab} = \begin{pmatrix} -e^{2Ar} & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad g^{ab} = \begin{pmatrix} -e^{-2Ar} & 0 \\ 0 & 1 \end{pmatrix}$$

General formula for the Christoffel symbols:

$$\Gamma^a{}_{bc} = \frac{1}{2} g^{ad} \left(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc} \right)$$

$$\Gamma^r{}_{rr} = \frac{1}{2} g^{rr} \partial_r g_{rr} = 0$$

$$\Gamma^r{}_{tr} = \Gamma^r{}_{rt} = \frac{1}{2} g^{rr} \left(\partial_t g_{rr} + \partial_r g_{tr} - \partial_r g_{tr} \right) = 0$$

$$\Gamma^r{}_{tt} = \frac{1}{2} g^{rr} \left(2 \partial_t g_{tr} - \partial_r g_{tt} \right) = A e^{2Ar}$$

$$\Gamma^t{}_{tt} = \frac{1}{2} g^{tt} \partial_t g_{tt} = 0$$

$$\Gamma^t{}_{tr} = \Gamma^t{}_{rt} = \frac{1}{2} g^{tt} \left(\partial_t \cancel{g_{tr}} + \partial_r g_{tt} - \partial_t \cancel{g_{tr}} \right) = A$$

$$\Gamma^t{}_{rr} = \frac{1}{2} g^{tt} \left(2 \partial_r g_{rt} - \partial_t g_{rr} \right) = 0$$

(10) The calculation was done in the lectures with the result:

$(r, \theta) = (x^1, x^2)$

$$\Gamma^1{}_{22} = \Gamma^r{}_{\theta\theta} = -r$$

$$\Gamma^2{}_{12} = \Gamma^2{}_{21} = \Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = \frac{1}{r}$$

The remaining components of the Christoffels vanish.

The geodesic equation is

$$\frac{d^2x^a}{d\lambda^2} + \Gamma^a{}_{bc}\frac{dx^b}{d\lambda}\frac{dx^c}{d\lambda} = 0$$

Denote $\dot{r} = \frac{dr}{d\lambda}$, $\ddot{r} = \frac{d^2r}{d\lambda^2}$, $\dot{\theta} = \frac{d\theta}{d\lambda}$, $\ddot{\theta} = \frac{d^2\theta}{d\lambda^2}$

Then, the components of the geodesic equation are:

$$\ddot{r} + \Gamma^r{}_{\theta\theta}\dot{\theta}^2 = \ddot{r} - r\dot{\theta}^2 = 0$$

$$\ddot{\theta} + \Gamma^\theta{}_{r\theta}\dot{r}\dot{\theta} + \Gamma^\theta{}_{\theta r}\dot{\theta}\dot{r} = \ddot{\theta} + \frac{2}{r}\dot{\theta}\dot{r} = 0$$

Notice that the second equation can be written as

$$\ddot{\theta} + \frac{2}{r}\dot{\theta}\dot{r} = \frac{1}{r^2}\frac{d}{d\lambda}\left(r^2\dot{\theta}\right) = 0$$

Hence, $r^2\dot{\theta} = L$ = constant

$$\Rightarrow \quad \dot{\theta} = \frac{L}{r^2}$$

Plugging this result into the first equation,

$$\ddot{r} - r\dot{\theta}^2 = \ddot{r} - r\left(\frac{L^2}{r^4}\right) = \ddot{r} - \frac{L^2}{r^3} = 0$$

$$r^3\ddot{r} = L^2 \quad \Rightarrow \quad r(\lambda) = \sqrt{\frac{L^2 + c_1^2(\lambda + c_2)^2}{c_1}}$$

$$\theta(\lambda) = \arctan\left(\frac{c_1(\lambda + c_2)}{L}\right)$$

where c_1 and c_2 are

where c_1 and c_2 are integration constants and we have set to 0 the integration constant in the equation for θ.

Recalling that
$$\cos(\arctan(x)) = \frac{1}{\sqrt{1+x^2}}$$
$$\sin(\arctan(x)) = \frac{x}{\sqrt{1+x^2}}$$

the Cartesian coordinates along the geodesics are:

$$x = r(\lambda)\cos\theta(\lambda) = \sqrt{\frac{L^2 + c_1^2(\lambda+c_2)^2}{c_1}}\ \frac{1}{\sqrt{1+\frac{c_1^2(\lambda+c_2)^2}{L^2}}} =$$
$$= \frac{L}{\sqrt{c_1}}$$

$$y = r(\lambda)\sin\theta(\lambda) = \sqrt{\frac{L^2 + c_1^2(\lambda+c_2)^2}{c_1}}\ \frac{\frac{c_1(\lambda+c_2)}{L}}{\sqrt{1+\frac{c_1^2(\lambda+c_2)^2}{L^2}}} =$$
$$= \sqrt{c_1}\,(\lambda+c_2)$$

These are straight lines.

(11)

a) $$ds^2 = -dt^2 + a(t)^2 (dx^2 + dy^2 + dz^2)$$

$$\mathcal{L} = -\dot{t}^2 + a(t)^2 (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

Use the Euler-Lagrange equations:

$$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^a}\right) - \frac{\partial \mathcal{L}}{\partial x^a} = 0$$

t - component:

$$\frac{\partial \mathcal{L}}{\partial t} = 2\, a\, a' (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{t}} = -2\dot{t} \quad ; \quad \frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{t}}\right) = -2\ddot{t}$$

$$\Rightarrow \quad \ddot{t} + a\, a' (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = 0$$

$$\Rightarrow \quad \Gamma^t{}_{xx} = \Gamma^t{}_{yy} = \Gamma^t{}_{zz} = a\, a'$$

x - component :

$$\frac{\partial \mathcal{L}}{\partial x} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = 2\, a(t)^2\, \dot{x} \Rightarrow \frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = 2\, a(t)^2\, \ddot{x} + 4\, a(t) a'(t)\, \dot{t}\, \dot{x}$$

$$\Rightarrow \quad \ddot{x} + 2\frac{a'}{a}\, \dot{t}\, \dot{x} = 0$$

$$\Rightarrow \quad \Gamma^x{}_{tx} = \Gamma^x{}_{xt} = \frac{a'}{a}$$

and by symmetry $\Gamma^y{}_{ty} = \Gamma^y{}_{yt} = \Gamma^z{}_{tz} = \Gamma^z{}_{zt} = \frac{a'}{a}$

b) We have to check whether V^a satisfies the geodesic equation: $V^b \nabla_b V^a = 0$

t) : $V^b \nabla_b V^t = V^b (\partial_b V^t + \Gamma^t{}_{bc} V^c)$

$$= V^t \partial_t V^t + V^t \cancelto{0}{\Gamma^t{}_{tc}} V^c$$

$$+ V^x \cancelto{0}{\partial_x V^t} + V^x \Gamma^t{}_{xc} V^c$$

$$= -\frac{K^2}{a(t)^3} a' + V^x \Gamma^t{}_{xx} V^x = -\frac{K^2}{a(t)^3} a' + (a a')\left(\frac{K}{a(t)^2}\right)^2$$

$$= 0$$

x) $V^b \nabla_b V^x = V^b(\partial_b V^x + \Gamma^x{}_{bc} V^c)$

$$= V^t(\partial_t V^x + \Gamma^x{}_{tx} V^x)$$
$$+ V^x(\cancel{\partial_x V^x}^{\,0} + \Gamma^x{}_{xt} V^t)$$
$$= V^t\left(-\frac{2K}{a(t)^3} a' + 2\frac{a'}{a}\frac{K}{a(t)^2}\right) = 0$$

y) $V^b \nabla_b V^y = V^b(\cancel{\partial_b V^y}^{\,0} + \Gamma^y{}_{bc} V^c) = 0$

$V^b \nabla_b V^z = 0$

c) $U_a = g_{ab} U^b = g_{at} = -\delta^t{}_a$

Thus,

$$K_{ab} V^a V^b = a(t)^2 (V_a V^a + (V^t)^2)$$
$$= a(t)^2\left(-\cancel{(V^t)^2} + a(t)^2\left(\frac{K}{a(t)^2}\right)^2 + \cancel{(V^t)^2}\right)$$
$$= K^2$$

which is a constant. Hence

$$V^a \nabla_a (K_{bc} V^b V^c) = V^a \nabla_a (K^2) = 0$$