Week 8 Stochastic Process $dF(x) \to dF(W_t)$

13.1 The 'usual' differential of a function $dF(x)$

$F(x)$ is a function $F: \mathbb{R} \to \mathbb{R}$, $F'(x)$ is continuous

Def 13.1 $dF(x) = F'(x)\,dx$

dx small dW_t

13.2 Ito's formula for $dF(W_t)$

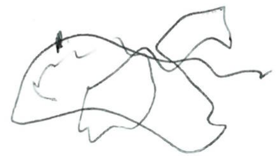
Q: What is $dF(W_t)$?

$g(x)$ is a differentiable function

$$dF(g(x)) = F'(g(x)) \boxed{g'(x)\,dx}$$

$$dF(W(t)) \neq F'(W(t)) \frac{dW(t)}{dt} dt$$

$\frac{dW(t)}{dt}$ does not exists



$$dF(g(x)) = F'(g(x)) \boxed{dg(x)}$$

randomly

$dg(x) = g'(x)\,dx$

cannot derivative anywhere

$$dF(W(t)) \neq F'(W(t))\,dW(t) + \cdots$$

Lemma 13.1

Ito's Lemma

$F'(x)$, $F''(x)$ $\quad$ $F''(x)$ is continuous

$$\cancel{dW_t} \quad dF(W_t) = F'(W_t)\,dW_t + \frac{1}{2}F''(W_t)\,dt \qquad (4)$$

$$dW_t = \Delta W_t = W(t+dt) - W(t)$$

Theorem 13.1

Suppose the $F(x)$ has two continuous and bounded derivatives:

$F'(x)$, $F''(x)$

$$F(W(b)) - F(W(a)) = \int_a^b F'(W_s)\,\underbrace{dW_s}_{(c)} + \frac{1}{2}\int_a^b F''(W_s)\,\underbrace{ds}_{(a)} \quad (5)$$

Corollary 13.1

$$\int_a^b F'(W_s)\,dW_s = F(W(b)) - F(W(a)) - \frac{1}{2}\int_a^b F''(W_s)\,ds \quad (6)$$

Eg 1. $\int_a^b 1\,dW_s$

$= \frac{dF(x)}{dx}$

$F(x) = x, \quad F'(x) = 1, \quad F''(x) = 0$

$X \to W_t \quad F(W_t) = W_t, \quad F'(W_t) = \frac{dF(W_t)}{dW_t} = 1, \quad F''(W_t) = 0$

$$\int_a^b \underbrace{F'(W_s)}_{=1}\,dW_s = \underbrace{\int_a^b dW_s}_{LHS} \overset{(6)}{=} W_b - W_a - 0 = W_b - W_a$$

Step 1: $F(x)$, $F'(x)$, $F''(x)$

Step 2: $X \to W_t$, $F(W_t)$, $F'(W_t)$, $F''(W_t)$

Step 3: Use (6)

$\frac{dW_t}{dt}$

Eg. 2 $F(x) = x^2$, $F'(x) = 2x$, $F''(x) = 2$ ← step 1

$F(W_t) = W_t^2$, $F'(W_t) = 2W_t$, $F''(W_t) = 2$ ← Step 2

Use (6)

$$\int_a^b F'(W_s)\,dW_s = \int_a^b 2W_s\,dW_s \overset{(6)}{=} F(W(b)) - F(W(a)) - \frac{1}{2}\int_a^b F''(W_s)\,ds$$ ← Step 3

$$= W_b^2 - W_a^2 - \frac{1}{2}\int_a^b 2\,ds$$

$$= W_b^2 - W_a^2 - (b-a)$$

$a=0$, $b=t$, $\int_0^t 2W_s\,dW_s = W_t^2 - 0^2 - (t-0)$

$$\boxed{\int_0^t \boxed{W_s}\,dW_s} = \frac{1}{2}W_t^2 - \frac{1}{2}t$$

$\int_0^t W_s^3\,dW_s$

$\int_0^t e^{W_s}\,dW_s$

$F'(x) = x$ $F'(W_s) = W_s$ $\frac{1}{2}W_t^2$ $F(x) = \frac{1}{2}x^2$

$F(x) = \frac{1}{2}x^2$ $F(W_s) = \frac{1}{2}W_s^2 + C$

Explanation of Ito's formula for $dF(W_t)$

$$F(x+dx) - F(x) = F'(x)\,dx + \frac{1}{2}F''(x)\,dx^2 + \frac{1}{3!}F^{(3)}(\theta)\,dx^3 \quad (7)$$

$$F(W_t + dW_t) - F(W_t) = F'(W_t)\,dW_t + \underbrace{\frac{1}{2}F''(W_t)\,dW_t^2}_{dt} + \frac{1}{3!}F^{(3)}(\theta)(dW_t)^3 \quad (8)$$

$$E(dW_t^2) = E\left((W_{t+dt} - W_t)^2\right) = \underbrace{\mathrm{var}(W_{t+dt} - W_t)}_{\sim N(0,\, dt)} + \Big[\underbrace{E(W_{t+dt} - W_t)}_{=0}\Big]^2$$

$$= dt$$

$$\boxed{dW_t^2 = dt} \quad (dW_t)^3 \sim (dt)^{\frac{3}{2}} < dt$$

$$dF(W_t) = F'(W_t)\,dW_t + \frac{1}{2}F''(W_t)\,dt \quad (4)$$

$$dW_t \sim \sqrt{dt} \quad 0.01$$

$$\vee$$

$$0.0001 \; dt$$

$F(W_t) \rightarrow F(t, W_t)$

13.2.2 Ito's formula for $F(t, W_t)$

$F(t, \underset{\substack{\uparrow \\ W_t}}{x})$ · function of t and x, $F: \mathbb{R}^2 \rightarrow \mathbb{R}$

Lemma 13.2 $dF(t, W_t)$

$$dF(t, W_t) = \left(\frac{\partial F(t, W_t)}{\partial t} + \frac{1}{2} \frac{\partial^2 F(t, W_t)}{\partial W_t^2} \right) \underline{dt} + \frac{\partial F(t, W_t)}{\partial W_t} \underline{dW_t}$$

Assume: all the derivatives $\underline{\text{exist}}$ $\underline{\text{continuous}}$

$$\frac{\partial F(t, W_t)}{\partial W_t} = \frac{\partial F(t, x)}{\partial x} \Big|_{x = W_t}, \quad \frac{\partial^2 F(t, W_t)}{\partial W_t^2} = \frac{\partial^2 F(t, x)}{\partial x^2} \Big|_{x = W_t}$$

E.g. $\boxed{F(t, x) = t^2 + x^2}$

$F(t, W_t) = t^2 + W_t^2$

① $\frac{\partial F}{\partial t} = 2t$, $\frac{\partial F}{\partial x} = 2x$, $\frac{\partial^2 F}{\partial x^2} = 2$ | ② $dF(t, \underline{x}) = (2t+1)dt + 2\underline{x}d\underline{x}$

③ $dF(t, W_t) = (2t + \frac{1}{2} \times 2)\, dt + \cancel{2x}\, \underbrace{2W_t\, dW_t}_{= 2x\, dx}$ $\qquad x = W_t$

$$= (2t + 1)\, dt + 2W_t\, dW_t$$

13.2.3 The chain rule

Y_t : stochastic process

$dY_t = a(t, Y_t)\, \underline{dt} + \sigma(t, Y_t)\, \underline{dW_t}$ $\qquad$ assume a, σ "good" functions

~~$F(t, W_t)$~~

Then $$dF(t, Y_t) = \left(\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial Y_t^2} + a \frac{\partial F}{\partial Y_t} \right) \boxed{dt} + \sigma \frac{\partial F}{\partial Y_t} \boxed{dW_t} \qquad (11)$$ chain rule

$$\frac{\partial F}{\partial t} = \frac{\partial F(t, Y_t)}{\partial t}, \quad \frac{\partial F}{\partial Y_t} = \frac{\partial F(t, Y_t)}{\partial Y_t}, \quad \frac{\partial^2 F}{\partial Y_t^2} = \frac{\partial^2 F(t, Y_t)}{\partial Y_t^2}$$

Ito's formula: $$dF(t, W_t) = \left(\frac{\partial F}{\partial t} + \frac{1}{2} \frac{\partial^2 F}{\partial W_t^2} \right) \boxed{dt} + \frac{\partial F}{\partial W_t} \boxed{dW_t}$$

$\Leftrightarrow$ Ito's formula for $F(t, W_t)$ is a special case of the Chain rule (11)

$a = 0$, $\sigma = 1$

Understand (11)

Taylor's formula:
$$dF(t, \underset{\downarrow \atop W_t}{x}) = \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial x} dx + \frac{1}{2}\frac{\partial^2 F}{\partial x^2} \underline{dx^2} + \frac{\partial^2 F}{\partial x \partial t} dx dt + \frac{1}{2}\frac{\partial^2 F}{\partial t^2} dt^2 \quad (12)$$

Fasts: $\boxed{\underline{dW_t^2} = \underline{dt}}$

$\underline{dt\, dW_t} \overset{\times}{=} dt \cdot \sqrt{dt} = \underline{(dt)^{\frac{3}{2}} < dt}$ $\quad \times \underline{(dt)^2} < dt$

delete all the terms of $\underset{\text{smaller}}{\underline{\text{lower order than}}}$ dt

(12):
$$dF(t, W_t) = \underline{\frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial W_t} dW_t + \frac{1}{2}\frac{\partial^2 F}{\partial W_t^2} \underbrace{dt}_{dW_t^2 = dt}} + \cancel{\frac{\partial^2 F}{\partial W_t \partial t} \underbrace{dW_t dt}_{< dt}} + \cancel{\frac{1}{2}\frac{\partial^2 F}{\partial t} \underbrace{dt^2}_{< dt}}$$

$$= \left(\frac{\partial F(t, W_t)}{\partial t} + \frac{1}{2}\frac{\partial^2 F(t, W_t)}{\partial W_t^2} \right) dt + \frac{\partial F(t, W_t)}{\partial W_t} dW_t$$

Ito's formula (11)

E.g: Replace x in (12)

$dY_t = a\, dt + 6\, dW_t$ $\leftarrow$ interet rate

$$(dY_t)^2 = a^2 \underbrace{dt^2}_{<dt} + 2a\sigma \underbrace{dt\,dW_t}_{<dt} + \sigma^2 \underbrace{dW_t^2}_{=dt}$$

$$= 0 + 0 + \sigma^2 dt$$

$$(dY_t)^2 = \sigma^2 dt$$

$$dF(t, Y_t) = \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial Y_t} dY_t + \frac{1}{2} \frac{\partial^2 F}{\partial Y_t^2} dY_t^2 + \cdots \underbrace{dt\,dY_t}_{<dt} + \cdots \underbrace{dt^2}_{<dt}$$

$$= \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial Y_t} (a\,dt + \sigma dW_t) + \frac{1}{2} \frac{\partial^2 F}{\partial Y_t^2} \cdot \sigma^2 dt$$

$$dF(t, Y_t) = \left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial Y_t} \cdot a + \frac{1}{2} \frac{\partial^2 F}{\partial Y_t^2} \sigma^2 \right) dt + \sigma \frac{\partial F}{\partial Y_t} dW_t$$

13.3 Stochastic differential equations (SDE)

Def. 13.2 SDE

$$dY_t = a(t, Y_t)\,dt + \sigma(t, Y_t)\,dW_t \qquad (14)$$

$a(t, Y_t)$, $\sigma(t, Y_t)$ are given random functions

and $Y_t = Y(t)$ unknow random process

Def 13.3 Solution of $Y(t)$

$$Y(t) = Y(0) + \int_0^t a(s, Y_s)\, ds + \int_0^t \sigma(s, Y_s)\, dW_s \quad (15)$$

$Y(t)$ solves (14) $\rightarrow$ diffusion process

$a(t, Y_t)$ drift

$\sigma(t, Y_t)$ volatility

Egs of SDEs

Eg 1. $dY_t = dW_t$, $Y(0) = 0$ $\qquad dY_t = 0\, dt + 1 \times dW_t$

$a = 0$

$\sigma = 1$

$$Y(t) = Y(0) + \int_0^t dW_s$$

$$= 0 + W_t - W_0 = W_t$$

$$Y(t) = W_t$$

Eg 2 $dY_t = \mu dt + \sigma dW_t$, $Y(0) = 1$ where μ and σ are constants

$$Y(t) = Y(0) + \int_0^t \mu\, ds + \int_0^t \sigma\, dW_s \quad \leftarrow (15)$$

$$Y(t) = \cancel{Y(0)}\ 1 + \mu t + \sigma(W_t - W_0)$$

$$Y(t) = 1 + \mu t + \sigma W_t$$

SDE for the price of a share

$$dS(t) = S(t+dt) - S(t)$$

$$dS(t) = \underbrace{S(t)\, a\,dt}_{\text{pushes price up if } a>0} + \underbrace{S(t) \cdot \xi(dt)}_{\text{a random noise}}, \qquad (16)$$

$\xi(dt)$: a random noise

pushes price up if $a > 0$

$\xi(dt) = \sigma dW_t$ ← choice

$$dS(t) = a\,S(t)\,dt + \sigma S(t)\,dW_t \qquad (17)$$

$$S(0) = S_0$$

Q: how to solve (17)? $S(t)$

Theorem 13.2 Solutions to Eq (17)

$$S_t = S_0 e^{(a - \frac{\sigma^2}{2})t + \sigma W_t}$$

Proof: Rewrite (17)

$$\underbrace{\frac{dS(t)}{S(t)}} = a\, dt + \sigma dW_t \quad \text{with } S(0) = S_0$$

$$d \ln S(t) = a\, dt + \sigma dW_t$$

Use the chain rule

$$dF(S_t) = F'(S_t) dS_t + \frac{1}{2} F''(S_t) (dS_t)^2$$

$$F(x) = \ln x; \quad F'(x) = (\ln x)' = \frac{1}{x}, \quad F''(x) = -\frac{1}{x^2}$$

$$\ln S(t) \qquad (\ln S(t))' = \frac{1}{S_t}, \quad (\ln S(t))'' = -\frac{1}{S_t^2}$$

$$(dS_t)^2 = [a S(t) dt + \sigma S(t) dW_t]^2 \quad \text{using (17)}$$

$$= a^2 S^2(t) (dt)^2 + 2a\sigma S^2(t) \, dt \cdot dW_t + \sigma^2 (S(t))^2 (dW_t)^2$$

$$dt \qquad \boxed{(dW_t)^2 = dt} \qquad (dt)^{\frac{3}{2}} < dt$$

$$= \sigma^2 S_t^2 \, dt$$

$$d \ln(S_t) = \frac{1}{S_t} (a S_t dt + \sigma S_t dW_t) + \frac{1}{2} \left(-\frac{1}{S_t^2}\right) \times \sigma^2 S_t^2 dt$$

(17)

$$d \ln(S_t) = \left(a \boxed{-\frac{\sigma^2}{2}}\right) dt + \sigma dW_t$$

note: $d \ln(S_t) \neq a dt + \sigma dW_t$

$$\int_0^t d \ln(S_t) = \int_0^t \left(a - \frac{\sigma^2}{2}\right) dt + \int_0^t \sigma \, dW_t$$

$$\ln S_t - \ln S_0 = \left(a - \frac{\sigma^2}{2}\right) t + \sigma W_t$$

$$\frac{S_t}{S_0} = e^{\left(a - \frac{\sigma^2}{2}\right) t + \sigma W_t} \qquad S_t = S_0 e^{\left(a - \frac{\sigma^2}{2}\right) t + \sigma W_t} \quad \square$$

Theorem 13.3

$$f(t, x) = S_0 e^{\mu t + \sigma x}, \quad \text{where } \mu = a - \frac{\sigma^2}{2} \text{ and } S_0 = S(0).$$

Proof: Step 1: Ito's lemma

$$\boxed{df(t, W_t)} = \left(\frac{\partial f(t, W_t)}{\partial t} + \frac{1}{2} \frac{\partial^2 f(t, W_t)}{\partial W_t^2} \right) dt + \frac{\partial f(t, W_t)}{\partial W_t} dW_t \quad (19)$$

$$dS(t) = aS(t)\,dt + \sigma S(t)\,dW_t \quad (17)$$

$$\boxed{df} = a f\,dt + \sigma f\,dW_t \qquad f = f(t, W_t)$$

$$\left(\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial W_t^2} \right) \boxed{dt} + \frac{\partial f}{\partial W_t} \boxed{dW_t} = a f \boxed{dt} + \sigma f \boxed{dW_t} \quad (20)$$

Equating the coeffs in front of dW_t in (20)

$$\frac{\partial f(t, \underset{x}{W_t})}{\partial \underset{x}{W_t}} = \sigma f(t, \underset{x}{W_t}) \overset{\text{notations}}{\Longleftrightarrow} f_x'(t, x) = \sigma f(t, x) \quad (22)$$

Fix t. (22) has the general solution:

$$f(t,x) = c(t)\, e^{\sigma x} \qquad (23) \qquad \text{Check it!}$$

Step 2: Find $c(t)$

Equating the coeffs in front of dt

$$f'_t(t,x) + \frac{1}{2} f''_{xx}(t,x) = a f(t,x) \qquad (24)$$

From (23): $\quad f'_t(t,x) = c'(t)\, e^{\sigma x}$

$$f''_{xx}(t,x) = c(t)\, \sigma^2 e^{\sigma x}$$

$$c'(t)\, e^{\sigma x} + \frac{1}{2}\sigma^2 c(t)\, e^{\sigma x} = a\, c(t)\, e^{\sigma x}$$

$$c'(t) = \left(a - \frac{\sigma^2}{2}\right) c(t) \qquad (27)$$

Fact: $y'(x) = \alpha y(x)$

then $y(x) = \underset{\downarrow}{c}\, e^{\alpha x}$

constant

$$c(t) = c_0\, e^{(a - \frac{\sigma^2}{2})t} \qquad \text{where } c_0 = c(0)$$

$$\boxed{f(t,x)} = c_0\, e^{\mu t + \sigma x}, \quad \text{where } \mu = a - \frac{\sigma^2}{2} \qquad \blacksquare$$

$$S(t) = f(t, W_t) = C_0 e^{\mu t + \sigma x}$$

Since $\underset{t=0}{S(0)} = C_0$, $C_0 = S_0$

$$S(t) = S_0 e^{\mu t + \sigma W_t}$$

The Ornstein - Uhlenbeck Process (OUP)

Definition 13.4 OUP

$r(t)$ is OUP if

$$dr = -a(r-\mu)\,dt + \sigma\, dW_t \qquad (30)$$

a, μ, σ

$\underbrace{a > 0, \quad \mu > 0}$, $\underline{\sigma > 0}$ optional

Case : $\sigma = 0$

$$dr = -a(r-\mu)\,dt$$

Since $dr = r'\,dt$

$$r' = -a(r-\mu) \tag{31}$$

$$(r-\mu)' = -a(r-\mu) \qquad \text{as } (r-\mu)' = r' - \mu' = r'$$

$$r - \mu = c e^{-at}$$

$$r(t) = \mu + c e^{-at}$$

$a > 0 \qquad e^{-at} \to 0$ as $t \to \infty$

$r(t) \to \mu$ when $t \to \infty$

Theorem 13.4 solution to OUP

$$dr = -a(r-\mu)\,dt + \sigma dW_t$$

Then $r(t) = b + (r(0)-\mu)e^{-at} + \sigma e^{-at}\int_0^t e^{as}\,dW_s$

Proof: Guess. $r(t) - \mu = u(t)\,e^{-at}$ (32)

Then $u(t) = e^{at}(r(t)-\mu)$;

If $u = f(t,r)$, then $du = \underline{f_t'}\,dt + \underline{f_r'}\,dr + \underline{\frac{1}{2}f_{rr}''(dr)^2}$ Eq (13)

$u(t) = f(t,r) \overset{(32)}{=} e^{at}(r-\mu)$

$f_t' = \frac{\partial}{\partial t}\left(e^{at}(r-\mu)\right) = a e^{at}(r-\mu)$

$f_r' = \frac{\partial}{\partial r}\left(e^{at}(r-\mu)\right) = e^{at}$

$f_{rr}'' = \frac{\partial}{\partial r}\left(e^{at}\right) = 0$

$$du(t) = a\,e^{at}(r-\mu)\,dt + e^{at}\,dr$$

$$= \underbrace{a e^{at}(r-\mu)\,dt} + e^{at}\big(\underbrace{-a(r-\mu)\,dt} + \sigma\,dW_t\big) \quad \text{def of } r(t) \text{ OUP}$$

$$= \sigma e^{at}\,dW_t$$

$$\int_0^t du(s) = \sigma \int_0^t e^{as}\,dW_s \qquad \text{integral on both sides}$$

$$u(t) - u(0) = \sigma \int_0^t e^{as}\,dW_s \qquad (33)$$

$$(32):\ r(0) - \mu = u(0) \quad \text{if } t=0$$

So (33) rewritten: $\boxed{u(t)} = \boxed{r(0) - \mu + \sigma \int_0^t e^{as}\,dW_s}$

Use (32) again, replace $u(t)$ $\cancel{\overset{\text{def}}{=} e^{at}(r(t)-\mu)}$ $\qquad r(t) - \mu = \boxed{u(t)}\,e^{-at}$ (32)

$$r(t) = \mu + e^{-at}\left(r(0) - \mu + \sigma \int_0^t e^{as}\,dW_s\right)$$

$$= r(0)e^{-at} + \mu(1 - e^{-at}) + \sigma e^{-at}\int_0^t e^{as}\,dW_s$$

$$= (r(0)-\mu)e^{-at} + \mu + \sigma e^{-at}\int_0^t e^{as}\,dW_s \qquad b = \mu \quad \boxed{14}$$