Queen Mary University of London

MTH5114 **Linear Programming and Games, Spring 2024**

Week 6 Seminar Questions **Viresh Patel**

Practice Questions: Solve the following linear programs using the 2-phase Simplex algorithm. For each question, state the initial basic feasible solution found by the first phase, and the final optimal basic feasible solution to the linear program, together with its objective value. If one of these solutions does not exist, explain why.

You should indicate the highlighted row and columns in each pivot step as well as the row operations you carry out. This is in order to gain credit (e.g. in an exam) even if the final answer is incorrect.

(a)

$$\begin{aligned} \text{maximize} \quad & 3x_1 + x_2 + 2x_3 \\ \text{subject to} \quad & -x_1 + x_2 - x_3 = 1, \\ & x_1 + x_2 + 2x_3 \geq 3, \\ & 2x_1 + x_2 + 2x_3 \leq 4, \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Solution: Reformulating in standard equation form gives us:

$$\begin{aligned} \text{maximize} \quad & 3x_1 + x_2 + 2x_3 \\ \text{subject to} \quad & -x_1 + x_2 - x_3 = 1, \\ & -x_1 - x_2 - 2x_3 + s_1 = -3, \\ & 2x_1 + x_2 + 2x_3 + s_2 = 4, \\ & x_1, x_2, x_3, s_1, s_2 \geq 0 \end{aligned}$$

We now need to add artificial variable into any equation without a slack variable and into any equation with a negative right-hand side. The artificial variable's sign should match the right-hand side. The objective for our phase 1 program is to minimise the sum of artificial variables (or, equivalently, to maximise, -1 times this sum).

$$\begin{aligned} \text{maximize} \quad & -a_1 - a_2 \\ \text{subject to} \quad & -x_1 + x_2 - x_3 + a_1 = 1, \\ & -x_1 - x_2 - 2x_3 + s_1 - a_2 = -3, \\ & 2x_1 + x_2 + 2x_3 + s_2 = 4, \\ & x_1, x_2, x_3, s_1, s_2, a_1, a_2 \geq 0 \end{aligned}$$

Now, we form a tableau:

	x_1	x_2	x_3	s_1	s_2	a_1	a_2	
a_1	-1	1	-1	0	0	1	0	1
a_2	-1	-1	-2	1	0	0	-1	-3
s_2	2	1	2	0	1	0	0	4
$-w$	0	0	0	0	0	-1	-1	0
$-z$	3	1	2	0	0	0	0	0

This is not yet valid, we need to multiply the a_2 row by -1 to make a_2's entry 1:

	x_1	x_2	x_3	s_1	s_2	a_1	a_2	
a_1	-1	1	-1	0	0	1	0	1
a_2	1	1	2	-1	0	0	1	3
s_2	2	1	2	0	1	0	0	4
$-w$	0	0	0	0	0	-1	-1	0
$-z$	3	1	2	0	0	0	0	0

Finally, we add the rows for the artificial variables into the $-w$ row. The first valid tableau is given by:

	x_1	x_2	x_3	s_1	s_2	a_1	a_2	
a_1	-1	1	-1	0	0	1	0	1
a_2	1	1	2	-1	0	0	1	3
s_2	2	1	2	0	1	0	0	4
$-w$	0	2	1	-1	0	0	0	4
$-z$	3	1	2	0	0	0	0	0

Now, we carry out the first phase:

	x_1	x_2	x_3	s_1	s_2	a_1	a_2		
a_1	-1	1	-1	0	0	1	0	1	1
a_2	1	1	2	-1	0	0	1	3	3
s_2	2	1	2	0	1	0	0	4	4
$-w$	0	2	1	-1	0	0	0	4	
$-z$	3	1	2	0	0	0	0	0	

	x_1	x_2	x_3	s_1	s_2	a_1	a_2		
x_2	-1	1	-1	0	0	1	0	1	
a_2	2	0	3	-1	0	-1	1	2	$\frac{2}{3}$
s_2	3	0	3	0	1	-1	0	3	1
$-w$	2	0	3	-1	0	-2	0	2	
$-z$	4	0	3	0	0	-1	0	-1	

	x_1	x_2	x_3	s_1	s_2	a_1	a_2	
x_2	$-\frac{1}{3}$	1	0	$-\frac{1}{3}$	0	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{5}{3}$
x_3	$\frac{2}{3}$	0	1	$-\frac{1}{3}$	0	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$
s_2	1	0	0	1	1	0	-1	1
$-w$	0	0	0	0	0	-1	-1	0
$-z$	2	0	0	1	0	0	-1	-3

Here the first phase ends. Our initial basic feasible solution for the 2nd phase will be $x_2 = \frac{5}{3}$, $x_3 = \frac{2}{3}$, $s_2 = 1$, and $x_1, s_1 = 0$. We now continue with the second phase:

	x_1	x_2	x_3	s_1	s_2		
x_2	$-\frac{1}{3}$	1	0	$-\frac{1}{3}$	0	$\frac{5}{3}$	
x_3	$\frac{2}{3}$	0	1	$-\frac{1}{3}$	0	$\frac{2}{3}$	1
s_2	1	0	0	1	1	1	1
$-z$	2	0	0	1	0	-3	

	x_1	x_2	x_3	s_1	s_2		
x_2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	2	
x_1	1	0	$\frac{3}{2}$	$-\frac{1}{2}$	0	1	
s_2	0	0	$-\frac{3}{2}$	$\frac{3}{2}$	1	0	0
$-z$	0	0	-3	2	0	-5	

	x_1	x_2	x_3	s_1	s_2	
x_2	0	1	0	0	$\frac{1}{3}$	2
x_1	1	0	1	0	$\frac{1}{3}$	1
s_1	0	0	-1	1	$\frac{2}{3}$	0
$-z$	0	0	-1	0	$-\frac{4}{3}$	-5

Here the second phase ends. We have found an optimal basic feasible solution $x_1 = 1, x_2 = 2$, $x_3, s_1, s_2 = 0$, with objective value 5.

(b)

$$\begin{aligned} \text{maximize} \quad & -3x_1 + x_2 \\ \text{subject to} \quad & x_1 + x_2 \geq 5, \\ & -x_1 + x_2 \geq 2, \\ & x_1 - 2x_2 = 1, \\ & x_1, x_2 \geq 0 \end{aligned}$$

Solution: After writing the program in standard equation form, we get:

$$\begin{aligned} \text{maximize} \quad & -3x_1 + x_2 \\ \text{subject to} \quad & -x_1 - x_2 + s_1 = -5, \\ & x_1 - x_2 + s_2 = -2, \\ & x_1 - 2x_2 = 1, \\ & x_1, x_2, s_1, s_2 \geq 0 \end{aligned}$$

We now add artificial variables, and formulate the phase 1 program:

$$\begin{aligned} \text{maximize} \quad & -a_1 - a_2 - a_3 \\ \text{subject to} \quad & -x_1 - x_2 + s_1 - a_1 = -5, \\ & x_1 - x_2 + s_2 - a_2 = -2, \\ & x_1 - 2x_2 + a_3 = 1, \\ & x_1, x_2, s_1, s_2, a_1, a_2, a_3 \geq 0 \end{aligned}$$

We now start to formulate our tableau:

	x_1	x_2	s_1	s_2	a_1	a_2	a_3	
a_1	-1	-1	1	0	-1	0	0	-5
a_2	1	-1	0	1	0	-1	0	-2
a_3	1	-2	0	0	0	0	1	1
$-w$	0	0	0	0	-1	-1	-1	0
$-z$	-3	1	0	0	0	0	0	0

We need to multiply the first two rows by -1 to make the entries for a_1 and a_2, respectively, 1:

	x_1	x_2	s_1	s_2	a_1	a_2	a_3	
a_1	1	1	-1	0	1	0	0	5
a_2	-1	1	0	-1	0	1	0	2
a_3	1	-2	0	0	0	0	1	1
$-w$	0	0	0	0	-1	-1	-1	0
$-z$	-3	1	0	0	0	0	0	0

Finally, we need to add the rows with artificial variables to the row for $-w$ to obtain our first valid tableau:

	x_1	x_2	s_1	s_2	a_1	a_2	a_3	
a_1	1	1	-1	0	1	0	0	5
a_2	-1	1	0	-1	0	1	0	2
a_3	1	-2	0	0	0	0	1	1
$-w$	1	0	-1	-1	0	0	0	8
$-z$	-3	1	0	0	0	0	0	0

We now carry out the first phase:

	x_1	x_2	s_1	s_2	a_1	a_2	a_3		
a_1	1	1	-1	0	1	0	0	5	5
a_2	-1	1	0	-1	0	1	0	2	
a_3	1	-2	0	0	0	0	1	1	1
$-w$	1	0	-1	-1	0	0	0	8	
$-z$	-3	1	0	0	0	0	0	0	

	x_1	x_2	s_1	s_2	a_1	a_2	a_3		
a_1	0	3	-1	0	1	0	-1	4	$\frac{4}{3}$
a_2	0	-1	0	-1	0	1	1	3	
x_1	1	-2	0	0	0	0	1	1	
$-w$	0	2	-1	-1	0	0	-1	7	
$-z$	0	-5	0	0	0	0	3	3	

	x_1	x_2	s_1	s_2	a_1	a_2	a_3	
x_2	0	1	$-\frac{1}{3}$	0	$\frac{1}{3}$	0	$-\frac{1}{3}$	$\frac{4}{3}$
a_2	0	0	$-\frac{1}{3}$	-1	$\frac{1}{3}$	1	$\frac{2}{3}$	$\frac{13}{3}$
x_1	1	0	$-\frac{2}{3}$	0	$\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{11}{3}$
$-w$	0	0	$-\frac{1}{3}$	-1	$-\frac{2}{3}$	0	$-\frac{1}{3}$	$\frac{13}{3}$
$-z$	0	0	$-\frac{5}{3}$	0	$\frac{5}{3}$	0	$\frac{4}{3}$	$\frac{29}{3}$

At this point, the first phase ends, since all entries in the $-w$ row are non-positive. Because $-w > 0$, the linear program must be *infeasible* and so we stop here.