Please fill in feedback questionnaire on QMplus!

Recap quiz

An LP

$$\max \quad \underline{c}^T\underline{x}$$
$$\text{sub to} \quad A\underline{x} = \underline{b}, \ \underline{x} \geq \underline{0}$$

is called unbounded if, for every $k \geq 0$ there exists a feasible solution $\underline{x}$ such that $\underline{c}^T\underline{x} \geq k$

Suppose we apply simplex to above LP and final tableau is

	x_1	x_2	s_1	s_2	
s_1	1	-3	1	0	3
s_2	0	-8	2	1	10
	0	11	-3	0	-9

What is the next step?

We conclude LP is unbounded since no positive entries in highlighted column

Outline for today.

- Simplex method and unbounded LPs.
- So far can use simplex to solve

$$\max \underline{c}^T \underline{x}$$
$$\text{sub to } A\underline{x} = \underline{b} \quad \underline{x} \geq \underline{0}$$

$\boxed{\text{when } \underline{b} \geq 0}$

- What to do when some entries of $\underline{b}$ are less than 0.
- Complications that arise with simplex

Simplex algorithm and unbounded LPs

Recall step 3(b) in simplex, if all entries of highlighted column are ≤ 0 (ignoring final row) then LP is unbounded.

Explanation by example

Given LP

$$\begin{aligned} \max \quad & 3x_1 + 2x_2 \\ \text{sub to} \quad & x_1 - 3x_2 \leq 3 \\ & -2x_1 - 2x_2 \leq 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

If we use simplex

$$\begin{aligned} \max \quad & 3x_1 + 2x_2 \\ \text{sub to} \quad & x_1 - 3x_2 + s_1 = 3 \\ & -2x_1 - 2x_2 + s_2 = 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Initial tableau

	x_1	x_2	s_1	s_2		
s_1	1	-3	1	0	3	$3/1 = 3$
s_2	-2	-2	0	1	4	—
	3	2	0	0	0	

		x_1	x_2	s_1	s_2		
$R_1' = R_1$	x_1	1	-3	1	0	3	$3/1 = 3$
$R_2' = R_2 + 2R_1$	s_2	0	-8	2	1	10	—
$R_f' = R_f - 3R_1$		0	11	-3	0	-9	

Since all entries in highlighted column are negative we know LP is unbounded.

Let's try to see why

	x_1	x_2	s_1	s_2	
x_1	1	-3	1	0	3
s_2	0	-8	2	1	10
	0	11	-3	0	-9

$3/1 = 3$

—

Final tableau tells us our original LP is equivalent to

$$\begin{array}{lrl} \text{maximise} & & 11x_2 - 3s_1 + 9 \\ \text{sub to} & x_1 - 3x_2 + s_1 & = 3 \\ & -8x_2 + 2s_1 + s_2 & = 10 \\ & x_1, x_2, s_1, s_2 \geq 0. & \end{array}$$

It also gives us a BFS $\begin{pmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 10 \end{pmatrix}$

with objective value 9.

We can increase x_2 as much as we like and keep the constraints satisfied by also increasing the basic variables (x_1 and s_2). This will increase the objective.

Find feasible solution whose objective value is ≥ 100 by modifying feasible solution above.

ans: keep $s_1 = 0$ Want $11x_2 - 3s_1 + 9 \geq 100$

Take e.g. $x_2 = 10$. Now use constraints to find x_1 and s_2. $x_1 - 3x_2 + s_1 = 3$

$\Rightarrow x_1 = 33$

Also $-8x_2 + 2s_1 + s_2 = 10$

$\Rightarrow s_2 = 90.$

so $\begin{pmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 33 \\ 10 \\ 0 \\ 90 \end{pmatrix}$ is feasible with objective value $11x_2 - 3s_1 + 9 = 119$.

$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 33 \\ 10 \end{pmatrix}$ is feasible with obj value 119 in original LP

(check).

2 – phase simplex algorithm

Given LP in standard inequality form

$$\max \underline{c}^T \underline{x} \qquad \underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad A \text{ is } m \times n \text{ matrix.}$$
$$\text{sub to } A\underline{x} \le \underline{b}, \quad \underline{x} \ge \underline{0}$$

If $\underline{b} \ge \underline{0}$, can implement simplex by first changing to standard equation form

Starting with BFS $\begin{pmatrix} x_1 \\ \vdots \\ x_n \\ s_1 \\ \vdots \\ s_m \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ b_1 \\ \vdots \\ b_m \end{pmatrix}$

This is feasible if $b_i \ge 0 \ \forall i$ and then we apply algorithm from last time.

If $b_i < 0$ for some i then can't use this (not feasible)

Have to do some work to find a starting feasible solution.

The problem of finding a feasible solution of an LP can itself be cast as a (different) LP.

Example

maximise $10x_1 + 15x_2 + 8x_3$
sub to
$$8x_1 + 6x_2 + 12x_3 \leq 24$$
$$4x_1 + 6x_2 + 6x_3 \geq 6$$
$$6x_1 + 4x_2 + 8x_3 = 12$$
$$x_1, x_2, x_3 \geq 0.$$

Transform to standard eqn form

(A) maximise $10x_1 + 15x_2 + 8x_3$
sub to
$$8x_1 + 6x_2 + 12x_3 + s_1 = 24$$
$$-4x_1 - 6x_2 - 6x_3 + s_2 = -6 \quad \text{problem}$$
$$6x_1 + 4x_2 + 8x_3 = 12 \quad \text{problem}$$
$$x_1, x_2, x_3, s_1, s_2 \geq 0$$

No easy starting feasible solution because of constraints 2 and 3.

Introduce artificial variables into constraints to artificially create an easy starting feasible solution.

(B) ~~min $a_1 + a_2$~~ max $-a_1 - a_2$

sub to
$$8x_1 + 6x_2 + 12x_3 + s_1 = 24$$
$$-4x_1 - 6x_2 - 6x_3 + s_2 - a_1 = -6 \quad \text{problem}$$
$$6x_1 + 4x_2 + 8x_3 + a_2 = 12 \quad \text{problem}$$
$$x_1, x_2, x_3, s_1, s_2, a_1, a_2 \geq 0$$

A feasible solution to (B) where $a_1 = a_2 = 0$ gives a feasible solution to (A)

1. Write a goal for (B) that would give a feasible solution for (A)
2. Give easy feasible solution for (B)

Ⓑ $\max\ -a_1 - a_2$

sub to

$$
\begin{aligned}
8x_1 + 6x_2 + 12x_3 + s_1 &= 24 \\
-4x_1 - 6x_2 - 6x_3 + s_2 - a_1 &= -6 \quad \text{problem} \\
6x_1 + 4x_2 + 8x_3 + a_2 &= 12 \quad \text{problem} \\
x_1, x_2, x_3, s_1, s_2, a_1, a_2 &\geq 0
\end{aligned}
$$

A feasible solution to Ⓑ where $a_1 = a_2 = 0$ gives a feasible solution to Ⓐ

So we try to minimise $a_1 + a_2$

i.e. maximise $-a_1 - a_2$.

Have easy starting feasible solution for Ⓑ

$(x_1, x_2, x_3, s_1, s_2, a_1, a_2)$

$= (0, 0, 0, 24, 0, 6, 12)$

Now apply simplex.

Apply simplex. Write initial tableau

	x_1	x_2	x_3	s_1	s_2	a_1	a_2	
s_1	8	6	12	1	0	0	0	24
a_1	-4	-6	-6	0	1	-1	0	-6
a_2	6	4	8	0	0	0	1	12
$-w$	0	0	0	0	0	-1	-1	0
$-z$	10	15	8	0	0	0	0	0

we carry along original objective so that tableau is in correct form once we've found our starting feasible solution (in second phase)

This is objective function $-a_1 - a_2$ that we want to maximise in the first phase

		x_1	x_2	x_3	s_1	s_2	a_1	a_2	
R_1	s_1	8	6	12	1	0	0	0	24
R_2	a_1	-4	-6	-6	0	1	-1	0	-6
R_3	a_2	6	4	8	0	0	0	1	12
R_w	$-w$	0	0	0	0	0	-1	-1	0
R_z	$-z$	10	15	8	0	0	0	0	0

Tableau not yet in valid form

Need to clear columns corresponding to basic variables (columns highlighted above).

Clear them in one step.

		x_1	x_2	x_3	s_1	s_2	a_1	a_2	
$R_1' = R_1$	s_1	8	6	12	1	0	0	0	24
$R_2' = -R_2$	a_1	4	6	6	0	-1	1	0	6
$R_3' = R_3$	a_2	6	4	8	0	0	0	1	12
$R_w' = R_w + R_3 - R_2$	$-w$	10	10	14	0	-1	0	0	18
$R_z' = R_z$	$-z$	10	15	8	0	0	0	0	0

Above tableau in valid form.

Now we apply simplex (from last week) remembering

R_w is our objective row

R_z carried along for convenience

	x_1	x_2	x_3	s_1	s_2	a_1	a_2		
s_1	8	6	12	1	0	0	0	24	$24/12 = 3$
a_1	4	6	6	0	-1	1	0	6	$6/6 = 1$
a_2	6	4	8	0	0	0	1	12	$12/8 = 3/2$
$-w$	10	10	14	0	-1	0	0	18	
$-z$	10	15	8	0	0	0	0	0	

$R_1' = R_1 - 12R_2'$

$R_2' = \frac{1}{6}R_2$

$R_3' = R_3 - 8R_2'$

$R_w' = R_w - 14R_2'$

$R_z' = R_z - 8R_2'$

	x_1	x_2	x_3	s_1	s_2	a_1	a_2	
s_1	0	-6	0	1	2	-2	0	12
x_3	2/3	1	1	0	-1/6	1/6	0	1
a_2	2/3	-4	0	0	4/3	-4/3	1	4
$-w$	2/3	-4	0	0	4/3	-7/3	0	4
$-z$	14/3	7	0	0	4/3	-4/3	0	-8

	x_1	x_2	x_3	s_1	s_2	a_1	a_2		
s_1	0	-6	0	1	2	-2	0	12	$12/2=6$
x_3	2/3	1	1	0	-1/6	1/6	0	1	—
a_2	2/3	-4	0	0	4/3	-4/3	1	4	$4/(\frac{4}{3})=3$
$-w$	2/3	-4	0	0	4/3	-7/3	0	4	
$-z$	14/3	7	0	0	4/3	-4/3	0	-8	

$R_1' = R_1 - 2R_3'$

$R_2' = R_2 + \frac{1}{6}R_3'$

$R_3' = \frac{3}{4}R_3$

$R_w' = R_w - \frac{4}{3}R_3'$

$R_z = R_z - \frac{4}{3}R_3'$

	x_1	x_2	x_3	s_1	s_2	a_1	a_2	
s_1	-1	0	0	1	0	0	-3/2	6
x_3	3/4	1/2	1	0	0	0	1/8	3/2
s_2	1/2	-3	0	0	1	-1	3/4	3
$-w$	0	0	0	0	0	-1	-1	0
$-z$	4	11	0	0	0	0	-1	-12

Here phase 1 ends. No positive values in R_w

Optimal solution for (B) given by $s_1 = 6$, $x_3 = 3/2$, $s_2 = 3$, and all other variables 0 (including a_1 and a_2)

This gives us a feasible solution for (A) namely

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3/2 \\ 6 \\ 3 \end{pmatrix}$$ (check)

Can now apply normal simplex to solve (A). Because we carried R_z our tableau is immediately ready to apply simplex.

Simply remove R_w and columns corresponding to a_1 and a_2 and apply pivot operations as usual.

Phase 2

	x_1	x_2	x_3	s_1	s_2	
s_1	-1	0	0	1	0	6
x_3	$3/4$	$1/2$	1	0	0	$3/2$
s_2	$1/2$	-3	0	0	1	3
$-z$	4	11	0	0	0	-12

	x_1	x_2	x_3	s_1	s_2	
s_1	-1	0	0	1	0	6
x_3	$\frac{3}{4}$	$\frac{1}{2}$	1	0	0	$\frac{3}{2}$
s_2	$\frac{1}{2}$	-3	0	0	1	3
$-z$	4	11	0	0	0	-12

$\dfrac{3/2}{1/2} = 3$

Pivot.

$R_1' = R_1$
$R_2' = 2R_2$
$R_3' = R_3 + 3R_2'$
$R_z' = R_z - 11R_2'$

	x_1	x_2	x_3	s_1	s_2	
s_1	-1	0	0	1	0	6
x_2	$\frac{3}{2}$	1	2	0	0	3
s_2	5	0	6	0	1	12
$-z$	$-\frac{25}{2}$	0	-22	0	0	-45

all entries ≤ 0
so stop.

optimal solution to A is $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 0 \\ 6 \\ 12 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$ is optimal solution to original LP

with objective value 45.

Q: We found optimal solution for (B) with objective value 0. (in phase 1).

what would it mean for (A) if instead

(i) (B) had optimal solution whose objective was not zero

means (A) is infeasible

(ii) (B) was infeasible (not possible (saw that (B) has an easy starting feasible solution).

(iii) (B) was unbounded

not possible. objective $= -a_1 - a_2 \leq 0$ because $a_1, a_2 \geq 0$.

(A)

$$
\begin{aligned}
\text{maximise} \quad & 10x_1 + 15x_2 + 8x_3 \\
\text{sub to} \quad & 8x_1 + 6x_2 + 12x_3 + s_1 = 24 \\
& -4x_1 - 6x_2 - 6x_3 + s_2 = -6 \\
& 6x_1 + 4x_2 + 8x_3 = 12 \\
& x_1, x_2, x_3, s_1, s_2 \geq 0.
\end{aligned}
$$

(B)

$$
\begin{aligned}
\max \quad & -a_1 - a_2 \\
\text{sub to} \quad & 8x_1 + 6x_2 + 12x_3 + s_1 = 24 \\
& -4x_1 - 6x_2 - 6x_3 + s_2 - a_1 = -6 \\
& 6x_1 + 4x_2 + 8x_3 + a_2 = 12 \\
& x_1, x_2, s_1, s_2, a_1, a_2 \geq 0
\end{aligned}
$$

Formal description of 2-phase simplex

① Given LP transform into standard equation form with slack variables just as before

② - If there is constraint with a slack variable s and $b<0$

say $$a_1x_1 + a_2x_2 + \cdots + a_nx_n + s = b$$

then replace with $$a_1x_1 + a_2x_2 + \cdots + a_nx_n + s - a = b$$

- If there is constraint with no slack variable

say $$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

then replace with $$a_1x_1 + a_2x_2 + \cdots + a_nx_n + a' = b$$

Here a, a' are artificial variables with sign restriction $a, a' \geq 0$
Each replaced constraint gets a different artificial variable.

③ Form initial tableau as before except

- <u>variables on the left</u>

  If a constraint has an artificial variable, put it on the left
  If a constraint has no artificial variable, put its slack variable on the left
  (Recall variables listed on the left are the basic variables in our current basic feasible solution)

- <u>Two rows below the line</u>

  We have a row R_w for our phase 1 objective which has a -1 for each artificial variable and zeros everywhere else

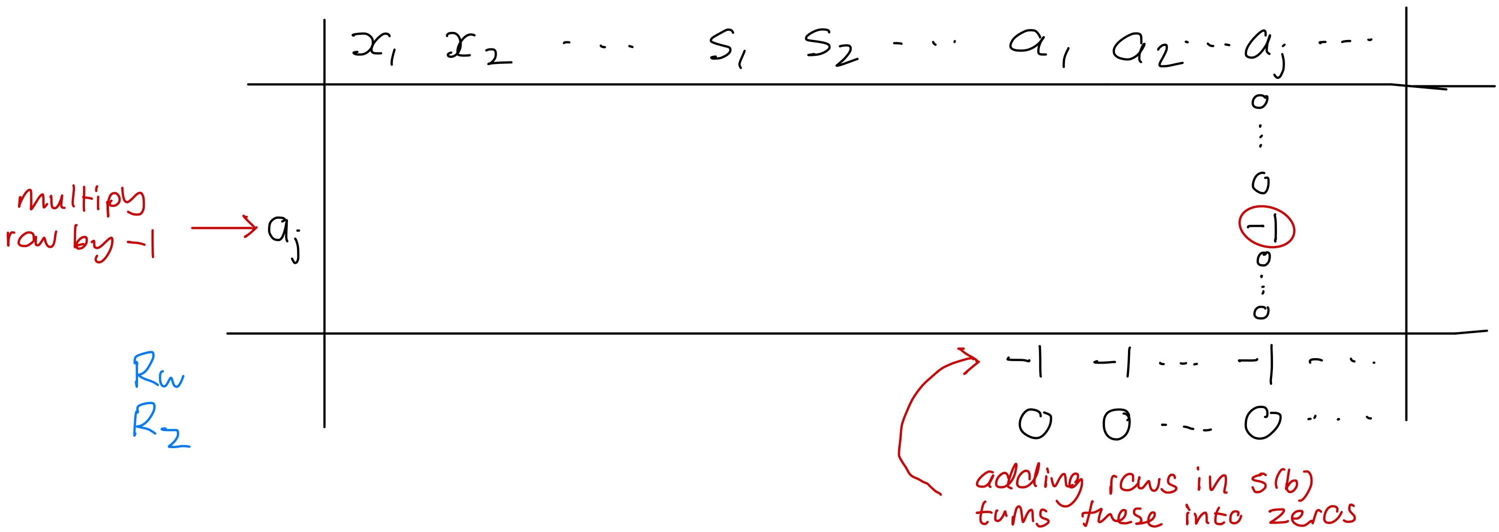
  We have a row R_z for our LP objective just as before.

④ Bring tableau into valid form so that each artificial variable has a single 1 and all other zeros in its column

Do this as follows:

(a) Ignoring R_z and R_w multiply rows by -1 where necessary so that each artificial variable has in its column (above the line) a single 1 (rather than -1)

(b) Every row with an artificial variable on the left is added to R_w (to remove -1's in columns of artificial variables).



⑤ Now apply standard simplex treating R_w as our objective.

(When clearing a column in a pivot, we also make sure we clear the column entry in R_z).

⑥ (a) If far right entry of R_w is 0 then have found our starting basic feasible solution.

Delete R_w and columns of artificial variables.

Apply standard simplex to this tableau with R_z as objective.

(We call this phase 2).

6 (a) If far right entry of R_w is 0 then have found our starting basic feasible solution.
Delete R_w and columns of artificial variables.
Apply standard simplex to this tableau with R_z as objective.
(We call this phase 2).

(b) If far right entry of R_w is > 0 then our original LP is infeasible.

<u>Note</u> It could happen in 6(a) that far right entry of R_w is zero, but some artificial variable is basic (i.e. appears on left).

Then we cannot immediately proceed to apply standard simplex.

Small, relatively easy step to deal with this but omitted here (and non-examinable).

Want you to get the main ideas and not be distracted by pathological situations.

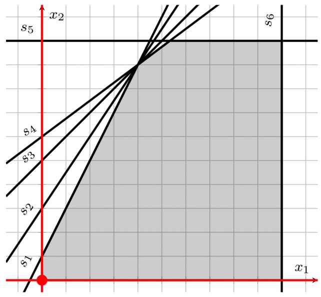
Does simplex algorithm always terminate?

- Sometimes a pivot operation does not change far right column, i.e. sometimes BFS stays unchanged and objective does not improve.

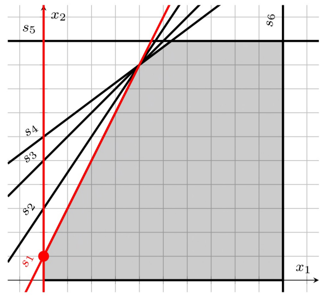
  Example on next page.

- When we apply the rules of simplex it can happen that we end up with exactly the same tableau we saw earlier! This is called cycling. (example next page)
- By adjusting the "tie-break" rules, we can avoid this and ensure simplex always terminates. We omit the details here.

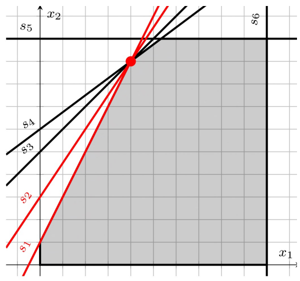
	x_1	x_2	s_1	s_2	s_3	s_4	s_5	s_6	
s_1	-2	1	1	0	0	0	0	0	1
s_2	$-\frac{3}{2}$	1	0	1	0	0	0	0	3
s_3	-1	1	0	0	1	0	0	0	5
s_4	$-\frac{3}{4}$	1	0	0	0	1	0	0	6
s_5	0	1	0	0	0	0	1	0	10
s_6	1	0	0	0	0	0	0	1	10
$-z$	0	1	0	0	0	0	0	0	0



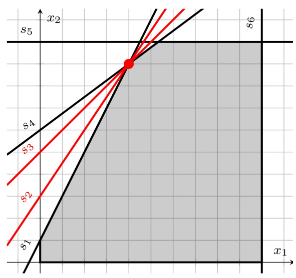
	x_1	x_2	s_1	s_2	s_3	s_4	s_5	s_6	
x_2	-2	1	1	0	0	0	0	0	1
s_2	$\frac{1}{2}$	0	-1	1	0	0	0	0	2
s_3	1	0	-1	0	1	0	0	0	4
s_4	$\frac{5}{4}$	0	-1	0	0	1	0	0	5
s_5	2	0	-1	0	0	0	1	0	9
s_6	1	0	0	0	0	0	0	1	10
$-z$	2	0	-1	0	0	0	0	0	-1



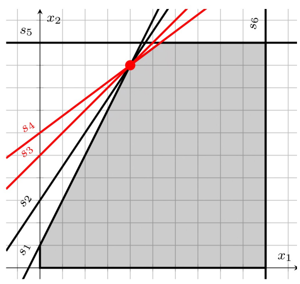
	x_1	x_2	s_1	s_2	s_3	s_4	s_5	s_6	
x_2	0	1	-3	4	0	0	0	0	9
x_1	1	0	-2	2	0	0	0	0	4
s_3	0	0	1	-2	1	0	0	0	0
s_4	0	0	$\frac{3}{2}$	$-\frac{5}{2}$	0	1	0	0	0
s_5	0	0	3	-4	0	0	1	0	1
s_6	0	0	2	-2	0	0	0	1	6
$-z$	0	0	3	-4	0	0	0	0	-9



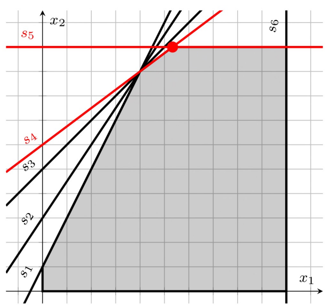
	x_1	x_2	s_1	s_2	s_3	s_4	s_5	s_6	
x_2	0	1	0	-2	3	0	0	0	9
x_1	1	0	0	-2	2	0	0	0	4
s_1	0	0	1	-2	1	0	0	0	0
s_4	0	0	0	$\frac{1}{2}$	$-\frac{3}{2}$	1	0	0	0
s_5	0	0	0	2	-3	0	1	0	1
s_6	0	0	0	2	-2	0	0	1	6
$-z$	0	0	0	2	-3	0	0	0	-9



	x_1	x_2	s_1	s_2	s_3	s_4	s_5	s_6	
x_2	0	1	0	0	-3	4	0	0	9
x_1	1	0	0	0	-4	4	0	0	4
s_1	0	0	1	0	-5	4	0	0	0
s_2	0	0	0	1	-3	2	0	0	0
s_5	0	0	0	0	3	-4	1	0	1
s_6	0	0	0	0	4	-4	0	1	6
$-z$	0	0	0	0	3	-4	0	0	-9



	x_1	x_2	s_1	s_2	s_3	s_4	s_5	s_6	
x_2	0	1	0	0	0	0	1	0	10
x_1	1	0	0	0	0	$-\frac{4}{3}$	$\frac{4}{3}$	0	$\frac{16}{3}$
s_1	0	0	1	0	0	$-\frac{8}{3}$	$\frac{5}{3}$	0	$\frac{5}{3}$
s_2	0	0	0	1	0	-2	1	0	1
s_3	0	0	0	0	1	$-\frac{4}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$
s_6	0	0	0	0	0	$\frac{4}{3}$	$-\frac{4}{3}$	1	$\frac{14}{3}$
$-z$	0	0	0	0	0	0	-1	0	-10



	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_5	0.5	−5.5	−2.5	9	1	0	0	0
x_6	0.5	−1.5	−0.5	1	0	1	0	0
x_7	1	0	0	0	0	0	1	1
$-z$	10	−57	−9	−24	0	0	0	0

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_1	1	−11	−5	18	2	0	0	0
x_6	0	4	2	−8	−1	1	0	0
x_7	0	11	5	−18	−2	0	1	1
$-z$	0	53	41	−204	−20	0	0	0

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_1	1	0	0.5	−4	−0.75	2.75	0	0
x_2	0	1	0.5	−2	−0.25	0.25	0	0
x_7	0	0	−0.5	4	0.75	−2.75	1	1
$-z$	0	0	14.5	−98	−6.75	−13.25	0	0

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_3	2	0	1	−8	−1.5	5.5	0	0
x_2	−1	1	0	2	0.5	−2.5	0	0
x_7	1	0	0	0	0	0	1	1
$-z$	−29	0	0	18	15	−93	0	0

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_3	−2	4	1	0	0.5	−4.5	0	0
x_4	−0.5	0.5	0	1	0.25	−1.25	0	0
x_7	1	0	0	0	0	0	1	1
$-z$	−20	−9	0	0	10.5	−70.5	0	0

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_5	−4	8	2	0	1	−9	0	0
x_4	0.5	−1.5	−0.5	1	0	1	0	0
x_7	1	0	0	0	0	0	1	1
$-z$	22	−93	−21	0	0	24	0	0

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_5	0.5	−5.5	−2.5	9	1	0	0	0
x_6	0.5	−1.5	−0.5	1	0	1	0	0
x_7	1	0	0	0	0	0	1	1
$-z$	10	−57	−9	−24	0	0	0	0