Theorem Let G be a graph, M a matching of G. Then M is a maximum matching of G if and only if there is <u>no</u> M-augmenting path in G.

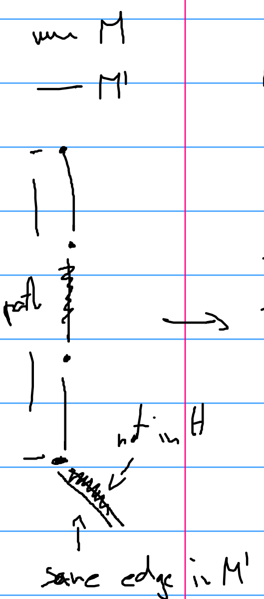
Proof. For the direction from left to right, assume that M is a maximum matching, and assume for contradiction that P is an M-augmenting path. Let $M' = M \triangle E(P)$. Then M' is a matching of G and $|M'| = |M| + 1$ a contradiction to maximality of M.

For the direction from right to left, assume that M is <u>not</u> a maximum matching of G. Let M' be a matching of G with $|M'| > |M|$. Let $S = M \triangle M'$ and let H be the graph with $V(H) = V(G)$ and $E(H) = S$. Each vertex $v \in V(H)$ is incident to at most two edges in S: $S \subseteq M \cup M'$, and in both M and M' there is at most one edge incident to v.

In other words $d_H(v) \leq 2$ for all $v \in V(H)$.

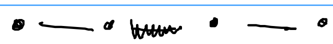
Each connected component of H is therefore either a path or a cycle.

Moreover, $d_H(v) = 2$ if and only if v is incident to one edge in M and another edge in M'. Therefore the paths and cycles in H <u>alternate</u> between M and M'.

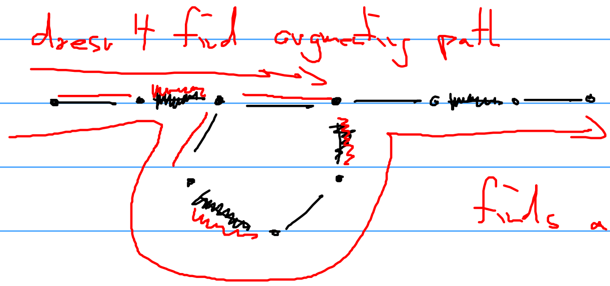


It follows in particular that H has even length. Since $|M'| > |M|$, there must be some connected component of H that is a path that <u>starts and ends with an edge in M'</u>.

→ The first and last vertex on that <u>path cannot be saturated</u> by M. Thus the path is an M-augmenting path. □



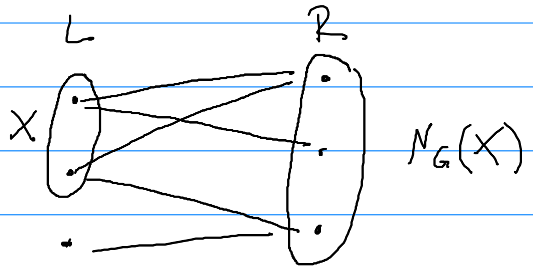
In graphs that are not bipartite, augmenting paths can be more difficult to find because of cycles of odd length



7.4 Saturating Matchings in Bipartite Graphs

__Theorem (Hall)__ Let G be a bipartite graph with parts L and R such that $|L| \leq |R|$. Then G has a matching that saturates L if and only if $|N_G(X)| \geq |X|$ for all $X \subseteq L$, where

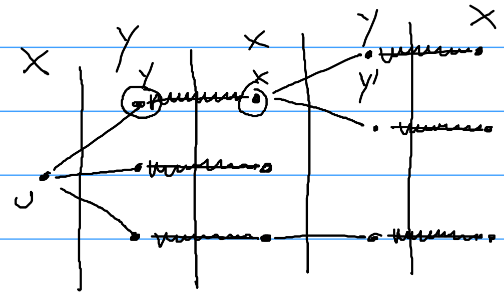
$$N_G(X) = \bigcup_{x \in X} N_G(x).$$



Proof. For the direction from left to right, assume that there exists a matching M that saturates L. Let $X \subseteq L$. Since every vertex in L is the endpoint of at most one edge in M, and every element of L is the endpoint of at least one edge in M, we can define a function $g: X \to N_G(X)$ that maps $x \in X$ to the other endpoint of the edge in M incident to x. Since every vertex in R is the endpoint of at most one edge in M, g is injective and therefore $|N_G(X)| \geq |X|$.
For the direction from right to left, assume that G does

not have a matching that saturates L. Let M be a maximum matching of G, and let $u \in L$ be a vertex not saturated by M.

Let $W = \{ x \in V(G) : G$ contains an M-alternating u-x-path$\}$

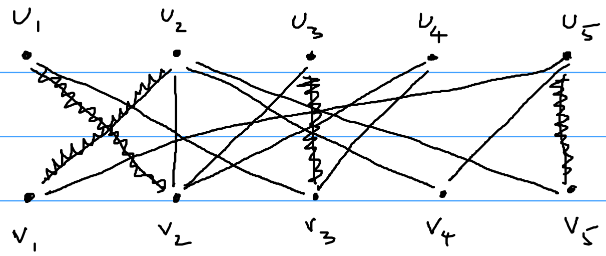


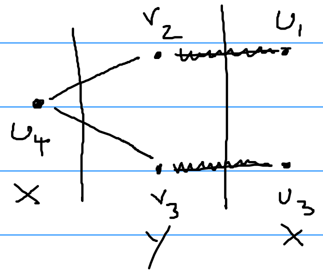
Let $X = W \cap L$ and $Y = W \cap R$.
Note that $u \in X$

By the previous theorem, G does not contain an M-augmenting path.
This means that for every $x \in X \setminus \{u\}$, there exists $y \in Y$ such that $xy \in M$, and for every $y \in Y$ there exists $x \in X \setminus \{u\}$ such that $xy \in M$. This defines a bijection between $X \setminus \{u\}$ and Y, so $|X \setminus \{u\}| = |Y|$, and $|X| = |Y| + 1$.

Consider $y \in N_G(X)$ and let $x \in X$ such that $xy \in E(G)$. If $xy \in M$, then y precedes x on any M-alternating u-x-path. If $xy \notin M$, then any M-alternating path that does not contain y can be turned into an M-alternating u-y-path by appending xy. In both cases, we conclude that $y \in W$ and therefore $y \in Y$. This means that $N_G(X) \subseteq Y$.

Therefore, $|N_G(X)| \leq |Y| = |X| - 1 < |X|$. $\square$



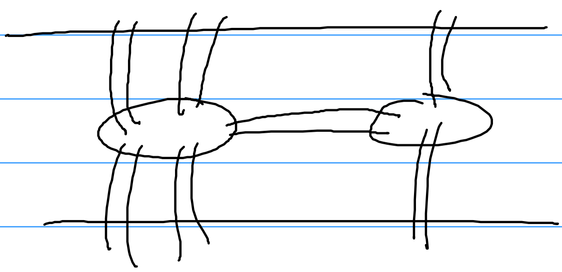


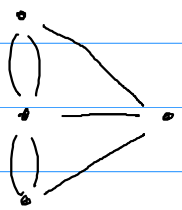
$X = \{u_1, u_3, u_4\}$

$N_G(X) = \{v_2, v_3\}$

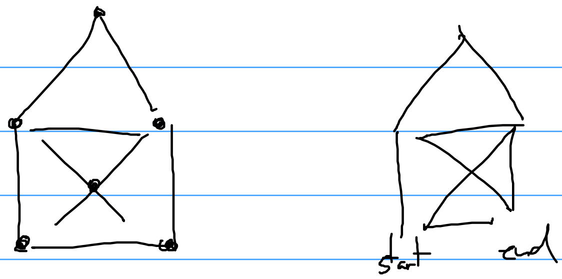
$|N_G(X)| < |X|$, so by Hall's theorem there is no matching that saturates L, so no matching of cardinality 5. The matching we have found has cardinality 4 and is a maximum matching.

8. Euler Trails and Tours





Definition An Euler trail is a trail that contains every edge of a graph. An Euler tour is an Euler trail that is closed.



<u>Theorem</u> Let G be a connected graph. Then G has an Euler tour if and only if $d_G(v)$ is even for all $v \in V(G)$.

Proof. For the direction from left to right, assume that G has an Euler tour R. Let $v \in V(G)$ and observe that the number of edges that occur in R immediately before v and the number of edges that occur in R immediately after v are the same. Since R contains every edge in $E(G)$ exactly once, $d_G(v)$ is equal to the number of edges that occur in R immediately before or after v, which is even.

For the direction from right to left, assume that $d_G(v)$ is even for all $v \in V(G)$. We show by induction on $|E(G)|$ that G has an Euler tour. First assume that $|E(G)| = 0$. By connectivity, $|V(G)| \leq 1$. If $|V(G)| = 0$, then the empty sequence is an Euler tour. If $|V(G)| = 1$, then the sequence containing the unique vertex is an Euler tour. Now assume that $|E(G)| > 0$, and that all connected graphs with fewer edges than G in which all degrees are even have an Euler tour. Since $|E(G)| > 0$ and since $d_G(v)$ is even for all $v \in V(G)$, G is not a tree and therefore contains a cycle. Let R be a tour of maximum length in G, and observe that $|E(R)| > 0$. Assume for contradiction that R is <u>not</u> an Euler tour. Let H be the graph with $V(H) = V(G)$ and $E(H) = E(G) \setminus E(R)$. Let J be a connected component of H with $|E(J)| > 0$. Then $d_J(v) = d_G(v) - d_R(v)$ for all $v \in V(J)$, which is the difference between two even numbers and therefore even. Thus J is a connected graph in which all vertices have even degrees, and

$|E(J)| < |E(G)|$, so by the induction hypothesis J has an Euler tour Q. Moreover since G is connected, there must exist a vertex $u \in V(J) \cap V(R)$. There thus exist a tour in G that starts at u, follows R and then follows Q. This tour has greater length than R, which is a contradiction. Therefore R must be an Euler tour. $\square$

<u>Corollary</u> Let G be a connected graph. Then G has an Euler trail if and only if $|\{v \in V(G) : d_G(v) \text{ is odd}\}| \leq 2$

Proof. For the direction from left to right, assume that there exists an Euler trail that starts at s and ends at t. If $s = t$, then the Euler trail is an Euler tour and by the theorem all degrees must be even. If $s \neq t$, then by the same argument as in the proof of the theorem, the degrees of all vertices except s and t must be even.

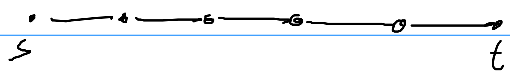
Now consider the direction from right to left. By Corollary 1.13, the number of vertices with odd degree is either 0 or 2. If it is 0, then by the theorem there exists an Euler tour, which is an Euler trail. Now assume that there are exactly two vertices $u, v \in V(G)$ with odd degrees.

Let H be the graph with $V(H) = V(G)$ and $E(H) = E(G) \cup \{e\}$, where e is a new edge with endpoints u and v. Then H is connected and $d_H(v)$ is even for all $v \in V(H)$, so H has an Euler tour by the theorem). If we arrange the tour so that it ends with edge e and vertex v and then remove these last two element, we obtain an Euler trail of G. $\square$

Algorithm Let G be a connected graph, $s,t \in V(G)$. Assume $d_G(v)$ is even for all $v \in V(G) \setminus \{s,t\}$. Let R be a maximal trail in G that starts at s, and repeat the following steps:

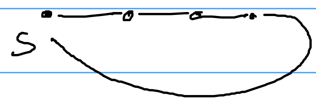
1. Let $x \in V(R)$ with $d_G(x) > d_R(x)$. If there is no such vertex, then stop: R is an Euler trail of G.
2. Let H be the graph with $V(H) = V(G)$ and $E(H) = E(G) \setminus E(R)$.
3. Let Q be a maximal trail in H that starts at x.
4. Replace one of the occurrences of x in R by Q.

- if s and t have odd degrees



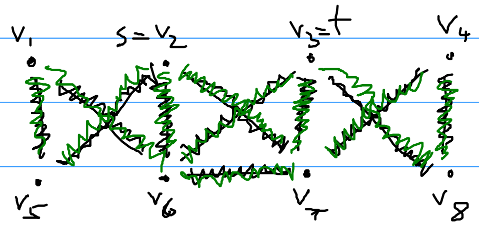
we can only get stuck at t

- if all degrees are even



we can only get stuck at s

Example



want to find an s-t-Euler trail

$v_2, v_6, v_7, v_4, v_8, v_3, v_7, v_2, v_5, v_1, v_6, v_3$

$v_2, v_6, v_3, \cancel{v_7}, v_4, v_8, v_3$

$\underbrace{}_{v_7, v_6, v_1, v_5, v_2, v_7}$