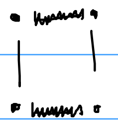
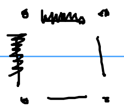
7 Matchings

a matching



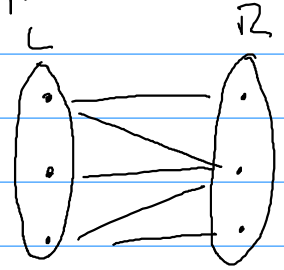


not a matching

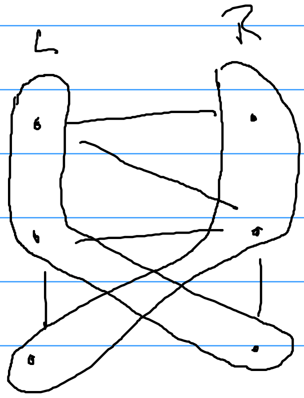
Definition Let G be a graph. A set $M \subseteq E(G)$ is a <u>matching</u> of G if every vertex of G appears at most once as an endpoint of an edge in M. A matching M of G is a <u>maximum matching</u> of G if it has maximum cardinality among all matchings of G. A matching M <u>saturates</u> $X \subseteq V(G)$ if every $x \in X$ is the endpoint of an edge in M. A matching M is a <u>perfect matching</u> if it saturates $V(G)$.

7.1 Bipartite Graphs

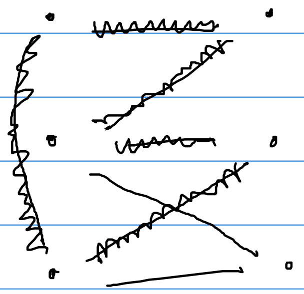
Definition Let G be a graph. Then G is <u>bipartite</u> if there exist $L, R \subseteq V(G)$ such that $L \cup R = V(G)$ and $L \cap R = \emptyset$, and every edge in $E(G)$ has one endpoint in L and one endpoint in R. Then L and R are called <u>parts</u> of G.



bipartite with parts L and R as drawn



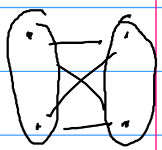
also bipartite, with L and R as drawn



not bipartite, because it contains a cycle of length 5, which is odd

<u>Theorem</u> Let G be a graph. Then G is bipartite if and only if it does <u>not</u> contain a cycle of odd length.

Proof. For the direction from left to right, assume that G is bipartite with parts L and R. Consider any cycle in G. All edges in G have one endpoint in L and one endpoint in R, so the cycle must alternate between L and R. Therefore the length of the cycle must be even.



For the direction from right to left, assume that G does not contain any cycles of odd length.

It suffices to show that each connected component of G is bipartite. Let H be a connected component of G, let T be a spanning tree of H, and $s \in V(H)$.

Let

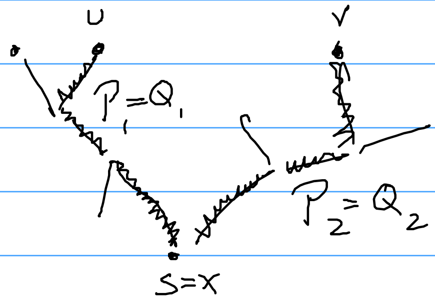
$$L = \{ v \in V(H) : \text{the unique } s\text{-}v\text{-path in } T \text{ has even length} \}$$

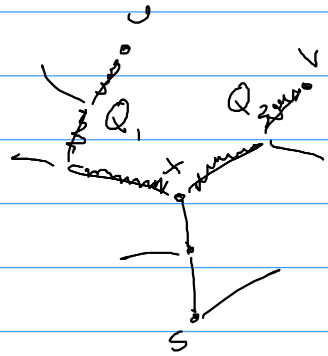
$$R = \{ v \in V(H) : \text{the unique } s\text{-}v\text{-path in } T \text{ has odd length} \}$$

Clearly $L \cup R = V(H)$ and $L \cap R = \emptyset$.

Assume for contradiction that G contains an edge uv such that $(u, v \in L)$ or $(u, v \in R)$.

Let P_1 be the unique s-u-path in T, and P_2 the unique s-v-path in T. Since P_1 and P_2 are both paths in T starting at s, they contain a common s-x-path for some $x \in V(T)$ (where possibly $x = s$) as well as an x-u-path Q_1 and an x-v-path Q_2 that don't share any edges.

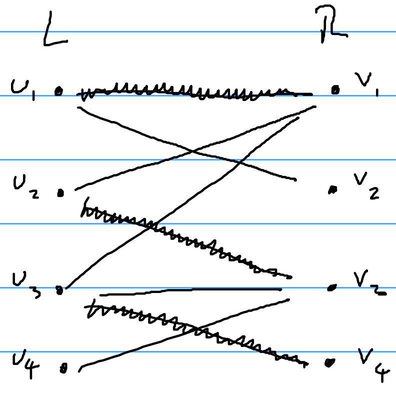




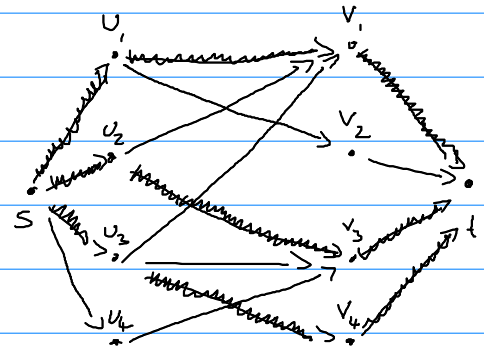
By definition of L and R, and by our choice of u and v, the lengths of P_1 and P_2 are either both even or both odd. The same is true for the lengths of Q_1 and Q_2 because they were obtained from P_1 and P_2 by removing the same s-x-path.

Together with the edge uv, Q_1 and Q_2 form a cycle, and the length of that cycle is odd. This is a contradiction. □

7.2 Maximum Matchings in Bipartite Graphs



G



(D_G, c_G)

matching M

$c_G(e) = 1$ for all $A(D_G)$

arcs carrying flow 1

Definition Let G be a bipartite graph with parts L and R. Then $\underline{(D_G, c_G)}$ is the directed network with

$$V(D_G) = V(G) \cup \{s,t\}$$

$$A(D_G) = \{uv \in E(G) : u \in L, v \in R\} \cup \{su : u \in L\} \cup \{vt : v \in R\}$$

and $c_G(e) = 1$ for all $e \in A(D_G)$.

Lemma Let G be a bipartite graph, M a matching of G. Let $f : A(D_G) \to \mathbb{R}$ such that $f(e) = 1$ if $e \in M$ or if one of the endpoints of e is equal to s or t, and the other endpoint is saturated by M. (otherwise $f(e) = 0$.) Then f is an s-t-flow of (D_G, c_G) and $|f| = |M|$.

Lemma Let f be an s-t-flow of (D_G, c_G) such that $f(e) \in \{0,1\}$ for all $e \in A(D_G)$. Let $M = \{uv \in E(G) : u \in L, v \in R, f(uv) = 1\}$. Then M is a matching of G and $|M| = |f|$.

Proof. For every $u \in L$,

$$|\{uv \in M : v \in R\}| = |\{uv \in E(G) : v \in R, f(uv) = 1\}|$$

$$\leq \sum_{e \in A^+_{D_G}(u)} f(e) \underset{\substack{\uparrow \\ \text{flow conservation} \\ \text{for } u}}{=} \sum_{e \in A^-_{D_G}(u)} f(e) = f(su) \underset{\substack{\nwarrow \\ \text{capacity of } su}}{\leq} 1$$

For every $v \in R$,

$$|\{uv \in M : u \in L\}| = |\{uv \in E(G) : u \in L, f(uv) = 1\}|$$

$$\leq \sum_{e \in A^-_{D_G}(v)} f(e) \underset{\substack{\uparrow \\ \text{flow conservation} \\ \text{for } v}}{=} \sum_{e \in A^+_{D_G}(v)} f(e) = f(vt) \underset{\substack{\nwarrow \\ \text{capacity of } vt}}{\leq} 1$$

Therefore M is a matching of G.

Moreover,

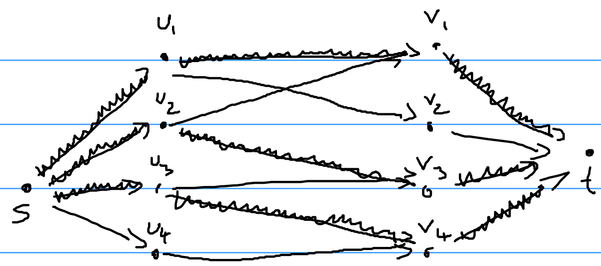
$$|M| = \{uv \in E(G) : u \in L, v \in R, f(uv) = 1\}$$
$$\underset{f \text{ integral}}{=} \sum_{u \in L} \sum_{e \in A^+_{D_G}} f(e) \underset{\substack{\text{flow}\\ \text{conservation for } u}}{=} \sum_{u \in L} \sum_{e \in A^-_{D_G}} f(e) = \sum_{u \in L} f(su) = |f| .$$

□

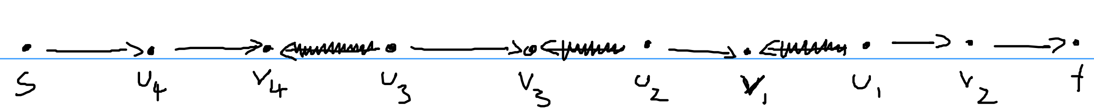
(D_G, c_G) has integer capacities, so the Ford-Fulkerson algorithm finds a maximum flow that is also integral, which using the second lemma corresponds to a matching M with $|M| = |f|$. There cannot exist a matching M' with $|M'| > |M|$: such a matching would imply, by the first lemma, the existence of a flow f' with $|f'| = |M'| > |M| = |f|$, contradicting the fact that f is a maximum flow. Thus M has to be a maximum matching.

7.3 Augmenting Paths (for matchings)

(D_G, c_G)

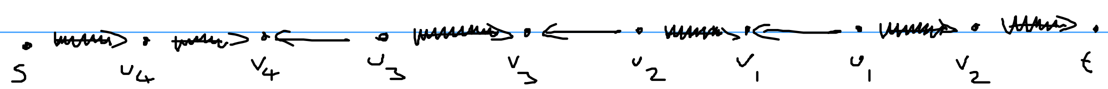


arcs on which flow is 1

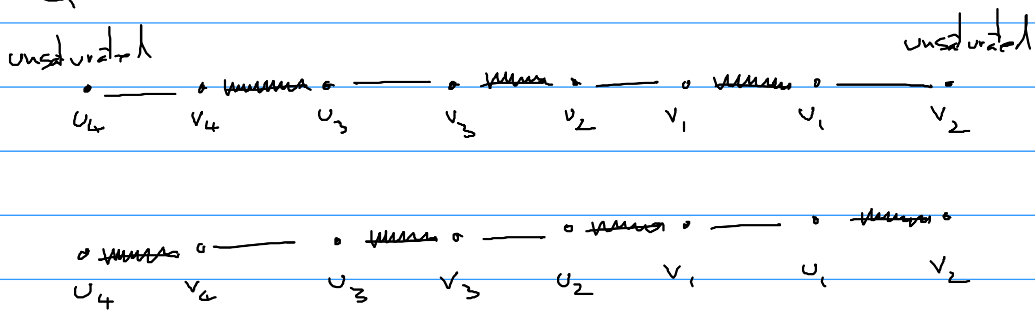


f-augmenting s-t-path in (D_G, c_G)

situation after sending one more unit of flow



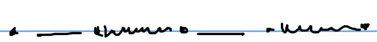
in G



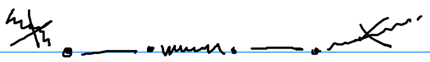
before

after

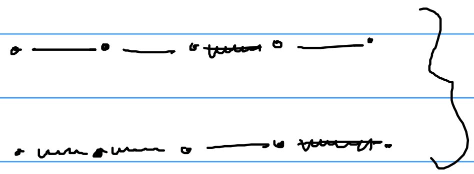
Definition Let G be a graph, M a matching of G. A path in G is an M-alternating path if it alternates between edges in M and edges in $E(G) \setminus M$, and an M-augmenting path if it is an M-alternating path and starts and ends with a vertex not saturated by M.



M-alternating but not M-augmenting



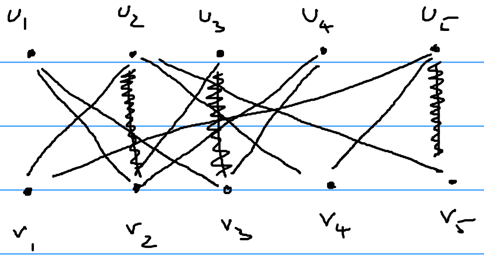
M-augmenting and thus also M-alternating



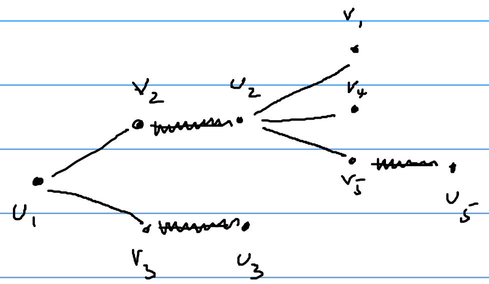
not M-alternating

We will use a variant of breadth-first search that starts from an unsaturated vertex and constructs a maximal tree of M-alternating paths. This tree will contain an M-augmenting path if there is one.

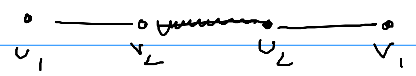
Example



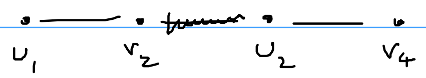
$\sim\!\sim$ M



the tree contains the following augmenting paths:



P_1

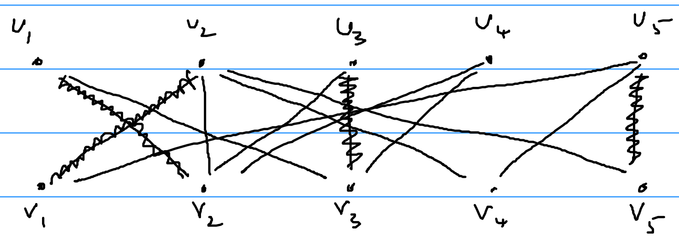


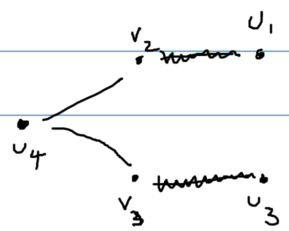
P_2

We use one of them, say ~~the first~~ P_1, to get a matching M' with $|M'| > |M|$

Let $M' = M \triangle E(P_1) = \{u_2v_2, u_3v_3, u_5v_5\} \triangle \{u_1v_2, u_2v_2, u_2v_1\}$

$= \{u_1v_2, u_2v_1, u_3v_3, u_5v_5\}$





maximal tree of alternating paths starting at u_4

no augmenting paths

Theorem Let G be a graph, M a matching of G. Then M is a maximum matching of G if and only if there is <u>no</u> M-augmenting path in G.