MTH6105 – Algorithmic Graph Theory **Spring 2024**

Problem Sheet 10 **F. Fischer**

You are expected to **attempt all exercises** before the seminar and to **actively participate** in the seminar itself.

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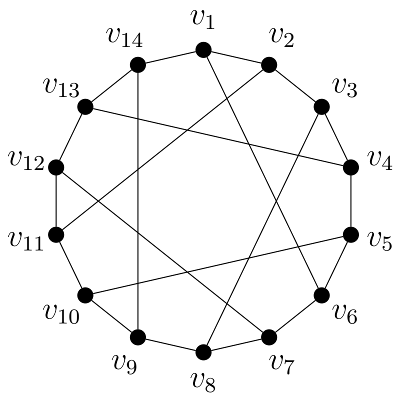
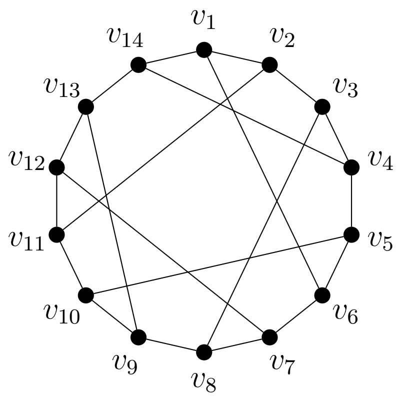
1. For each of the following two graphs, determine if the graph is bipartite. Justify your answer.





Solution: It is straightforward to verify that the graph on the left is bipartite with parts $L = \{v_1, v_3, v_5, v_7, v_9, v_{11}, v_{13}\}$ and $R = \{v_2, v_4, v_6, v_8, v_{10}, v_{12}, v_{14}\}$. The graph on the right is not bipartite because it contains the cycle $v_1v_2v_3v_4v_{14}v_1$, which has odd length.

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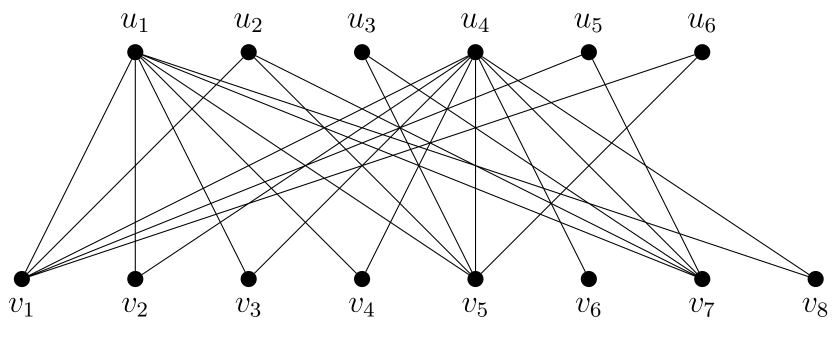
2. Consider the following bipartite graph G.



(a) Show that $M = \{u_1v_1, u_2v_5, u_3v_7, u_4v_2\}$ is a matching of G.

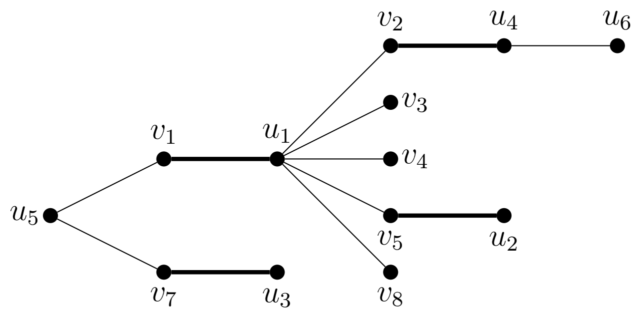
(b) Give an M-augmenting path of G.

(c) Give a maximum matching of G.

Solution:

(a) M is a matching of G because $M \subseteq E(G)$ and no vertex is an endpoint of more than one edge in M.

(b) Let $L = \{u_1, u_2, u_3, u_4, u_5, u_6\}$, and let $S = \{u_1, u_2, u_3, u_4, v_1, v_2, v_5, v_7\}$ be the set of vertices saturated by M. For each $x \in L \setminus S = \{u_5, u_6\}$, we can construct a maximal M-alternating tree with root x. Such a tree with root u_5 for example looks as follows.



Let $T = (V(G) \setminus S) \setminus \{u_5\}) = \{v_3, v_4, v_6, v_8\}$ be the set of vertices apart from u_5 that are contained in the tree with root u_5 and are not saturated by M. Then, for each $t \in T$, the unique u_5–t-path in this tree is an M-augmenting path. In particular, the path $P = u_5, v_1, u_1, v_3$ is an M-augmenting path.

(c) P is an M-augmenting path, so

$$\begin{aligned} M' = M \triangle E(P) &= \{u_1v_1, u_2v_5, u_3v_7, u_4v_2\} \triangle \{u_5v_1, u_1v_1, u_1v_3\} \\ &= \{u_1v_3, u_2v_5, u_3v_7, u_4v_2, u_5v_1\} \end{aligned}$$

is a matching of G with cardinality $|M'| = |M| + 1 = 5$. To show that M' is a maximum matching, we could repeat the procedure we used in Part (b) and conclude that there is no M'-augmenting path. As any matching of cardinality greater than that of M' would have to saturate L, we can also use Hall's theorem to argue that such a matching cannot exist. Let

$$\begin{aligned} U &= \{u \in L : u \text{ not saturated by } M'\} = \{u_6\} \quad \text{and} \\ X &= \{x \in L : \text{there exists an } M'\text{-alternating } u\text{–}x\text{-path in } G, \text{ where } u \in U\} \\ &= \{u_2, u_3, u_5, u_6\} \end{aligned}$$

Then $|N(X)| = |\{v_1, v_5, v_7\}| = 3 < 4 = |X|$, so by Hall's theorem G does not have a matching that saturates L. The cardinality of any matching of G is therefore at most 5, which means that M' is a maximum matching of G.

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3. A graph G is called *k-regular*, for $k \in \mathbb{N}$, if $d_G(v) = k$ for all $v \in V(G)$. Let G be a k-regular bipartite graph G with parts L and R.

   (a) Show that $|L| = |R|$.

   (b) Show that the directed network (D_G, c_G) has an s–t-flow of size $|L| = |R|$.

   (c) Show that G has a perfect matching.

Solution:

(a) We have that

$$|L| = 1/k \sum_{u \in L} d_G(u) = 1/k \sum_{v \in R} d_G(v) = |R|,$$

where the first and third equalities holds because G is k-regular and the second equality because G is bipartite.

(b) Let $f : E(D_G) \to \mathbb{R}$ such that for all $u \in L$ and $v \in R$, $f(su) = 1$, $f(uv) = 1/k$, and $f(vt) = 1$. It is straightforward to verify that f is an s–t-flow of (D_G, c_G) and $|f| = |L|$.

(c) Since $c_G(e) \in \mathbb{N}$ for all $e \in E(D_G)$, it follows from Theorem 6.16 that (D_G, c_G) has a maximum s–t-flow g such that $g(e) \in \mathbb{N}$ for all $e \in E(D_G)$. In particular, for such a flow, $g(e) \in \{0, 1\}$ for all $e \in E(D_G)$. By Lemma 7.7, G has a matching M with $|M| = |g| \geq |f| = |L|$, where the inequality holds because f is an s–t-flow of (D_G, c_G) and g is a maximum flow of (D_G, c_G), and the second equality by Part (b). The only way for M to have cardinality $|L|$ is for it to saturate both L and R, so M must be a perfect matching.